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Regularized finite element discretizations of a grade-two fluid model

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SUMMARY

We consider a system with three unknowns in a two-dimensional bounded domain which models the flow of a grade-two non-Newtonian fluid. We propose to compute an approximation of the solution of this problem in two steps: addition of a regularization term, finite element discretization of the regularized problem. We prove optimal *a priori* and *a posteriori* error estimates and present some numerical experiments. Copyright © 2005 John Wiley & Sons, Ltd.

RÉSUMÉ

Nous considérons un système à trois inconnues dans un domaine borné de dimension 2 qui modélise l'écoulement d'un fluide non newtonien de grade 2. Nous proposons de calculer une approximation de la solution de ce problème en deux étapes: addition d'un terme de régularisation, discrétisation par éléments finis du problème régularisé. Nous démontrons des estimations d'erreur *a priori* et *a posteriori* optimales et présentons quelques expériences numériques.

KEY WORDS: non-Newtonian fluids; regularization method; finite elements

1. INTRODUCTION

In a connected bounded open set Ω in $\mathbb{R}^2$ with a Lipschitz-continuous boundary, we consider the following system, called grade-two fluid model:

$$\begin{aligned} &-\nu\Delta\mathbf{u} + \mathbf{grad}\, p + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \\ &\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega \\ &\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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where

- for a vector field $\mathbf{v}$ with components v_1 and v_2, $\operatorname{curl}\mathbf{v}$ is equal to the scalar function $\partial_{x_1} v_2 - \partial_{x_2} v_1$,
- for a scalar function t and a vector field $\mathbf{v}$, $t \times \mathbf{v}$ denotes the vector field with components $-tv_2$ and tv_1.

Here, the parameters α and v are real constants, with $v > 0$, representing respectively the normal stress modulus and the viscosity (both divided by the constant density), and $\mathbf{f}$ denotes a density of body forces, usually proportional to the gravity acceleration. The unknowns are the velocity $\mathbf{u}$ and a modified pressure p. This non-Newtonian fluid model was introduced in Reference [1], and it can be noted that, if α is equal to zero, problem (1) reduces to the standard Navier–Stokes system, provided that p is replaced by $p - \frac{1}{2}|\mathbf{u}|^2$. The aim of this paper is to present and analyse a finite element discretization of this model.

Originally, the grade-two fluid model was not meant to apply to a specific material but it was rather a theoretical model intended to describe several non-Newtonian characteristic behaviours. It has been extensively studied for a long time, and, besides [1], we refer among others to References [2–7] for the main results of analysis (see also References [8–10] and the references therein). A similar but more complex model with application to an ice flow has been studied in Reference [11], and recently the equations for a time-dependent version of this model with $v = 0$ have been considered in Reference [12] for handling turbulent flows.

From a numerical point of view, discretizing problem (1) with simple Lagrange finite elements seems hopeless because the non-linear term involves derivatives of order three. Therefore following References [4, 5, 13] we introduce the auxiliary variable

$$z = \operatorname{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})$$

So, the first equation in (1) becomes

$$-v\Delta\mathbf{u} + \mathbf{grad}\, p + z \times \mathbf{u} = \mathbf{f} \quad \text{in } \Omega$$

Nevertheless, this equation is not sufficient to derive adequate *a priori* estimates. So, assuming that the function $\mathbf{f}$ has a square-integrable curl (which is always satisfied for instance when $\mathbf{f}$ is either the gravity or Coriolis acceleration), we take the curl of the equation above and this leads to the following transport equation:

$$vz + \alpha\mathbf{u} \cdot \nabla z = \alpha \operatorname{curl}\mathbf{f} + v \operatorname{curl}\mathbf{u} \quad \text{in } \Omega$$

where $\mathbf{u} \cdot \nabla z$ denotes the quantity $u_1 \partial_{x_1} z + u_2 \partial_{x_2} z$. Note that no boundary condition is imposed on z since they are given by $\mathbf{u}$ in the transport equation. Hence, we obtain the following system

$$\begin{aligned} &-v\Delta\mathbf{u} + \mathbf{grad}\, p + z \times \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \\ &\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \\ &vz + \alpha\mathbf{u} \cdot \nabla z = \alpha \operatorname{curl}\mathbf{f} + v \operatorname{curl}\mathbf{u} \quad \text{in } \Omega \\ &\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

Conversely, it is checked in Reference [13, Section 1] that, for each $(\mathbf{u}, z, p)$ satisfying (2), $(\mathbf{u}, p)$ is a solution of system (1) and z is equal to $\operatorname{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})$. The splitting proposed in (2) leads to a formulation that is much easier to handle, both from the theoretical and numerical points of view. In particular, its analysis does not require that the velocity has a bounded gradient.

The main properties of formulation (2) have been investigated in Reference [13], where its equivalent variational formulation is written. Moreover the existence of a solution is proven. However it seems that not so much work has been done concerning its discretization, apart from Reference [14]. Our choice of the discretization relies on the fact that the third line in (2) is a transport equation and that the solution of such a problem is known to have very weak regularity properties, i.e. that the gradient of its solution is not square-integrable in the general case. So the main idea is to add a regularization term to it, namely the term $-\varepsilon\Delta z$ for a positive small parameter ε, as first proposed in Reference [15]. We perform the analysis of the regularized problem: We prove the existence of a solution and a convergence result when ε tends to zero, together with an *a posteriori* estimate of the error induced by the regularization which gives an idea of the convergence order. Next, we propose a finite element discretization of the regularized problem, relying on a standard pair of finite elements for the Stokes problem for the $\mathbf{u}$ and p unknowns. Several consistent approximations of the z unknown are possible, we choose the simplest one. We check that the corresponding discrete problem has a solution in the neighbourhood of any solution of the regularized problem which is non-singular, in the sense that the linearized problem around this solution is well-posed. We also prove optimal *a priori* error estimates in appropriate norms.

In a final step, we perform the *a posteriori* analysis of the problem. Note that this analysis is now standard for the Stokes problem, even with a nonlinear term as in the first two lines of (2), see Reference [16] for instance. But not so much work has been done concerning the transport equation which appears in the third line of (2). We refer to Reference [17] for very interesting estimates concerning a simple transport equation, however with a stabilization term different from ours, and also to Reference [18] for recent results on a convection–diffusion equation with small diffusion coefficient. Here, we exhibit two types of error indicators, related to the addition of the regularization term and to the finite element discretization, and we prove nearly optimal *a posteriori* estimates. The aim of this part is to provide a tool first to optimize the choice of the parameter ε with respect to the finite element mesh and second to perform mesh adaptivity in order to increase the efficiency of the algorithm. We consider two iterative algorithms for solving the discrete problem: both of them are semi-implicit but, as first proposed in Reference [19], the second one involves a further upwind treatment of the transport term, which is well-known to enhance the convergence. Numerical experiments are then described; they are consistent with the analysis and we think that they justify the choices we make in this paper.

An outline of the paper is as follows:

In Section 2, we recall from Reference [13] the properties of the continuous problem. Section 3 is devoted to the analysis of the regularized problem and to the proof of estimates between the solutions of the initial and regularized problems. In Section 4, we propose a discrete problem, relying on finite element conforming approximation of the three unknowns and built from the variational formulation of the regularized problem by the Galerkin method. We prove optimal *a priori* error estimates. In Section 5, we describe the two types of error indicators and prove *a posteriori* estimates and upper bounds for the indicators. Section 6 is devoted to the description of the iterative algorithms that are used to solve the discrete problem and to the presentation of numerical tests.

2. THE CONTINUOUS PROBLEM

In what follows, for $1 < p < \infty$, we denote by $L^p(\Omega)$ the space of measurable real-valued functions v such that $|v|^p$ is integrable, with obvious extension to the case $p = \infty$. We also consider the corresponding Sobolev spaces $W^{m,p}(\Omega)$ for any nonnegative integer m. We introduce the subspace $L_0^2(\Omega)$ of functions in $L^2(\Omega)$ which have a zero integral on Ω. For any nonnegative real number s, we need the Hilbert Sobolev spaces $H^s(\Omega)$, provided with the norm $\|\cdot\|_{H^s(\Omega)}$ and semi-norm $|\cdot|_{H^s(\Omega)}$. As usual, $H_0^s(\Omega)$ stands for the closure in $H^s(\Omega)$ of the space $\mathscr{D}(\Omega)$ of infinitely differentiable functions with a compact support in Ω. Finally, we consider the space

$$H(\mathrm{curl}, \Omega) = \{\mathbf{g} \in L^2(\Omega)^2;\ \mathrm{curl}\,\mathbf{g} \in L^2(\Omega)\} \tag{3}$$

equipped with the natural norm

$$\|\mathbf{g}\|_{H(\mathrm{curl},\Omega)} = (\|\mathbf{g}\|^2_{L^2(\Omega)^2} + \|\mathrm{curl}\,\mathbf{g}\|^2_{L^2(\Omega)})^{1/2}$$

Assume that the data $\mathbf{f}$ belong to $H(\mathrm{curl}, \Omega)$. We consider the variational problem:

$$\begin{aligned}
&\text{Find } (\mathbf{u}, p, z) \text{ in } H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega) \text{ such that}\\
&\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + A(z; \mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x}\\
&\forall q \in L_0^2(\Omega), \quad b(\mathbf{u}, q) = 0\\
&\forall t \in L^2(\Omega), \quad c(z, t) + C(\mathbf{u}; z, t) = \alpha \int_\Omega (\mathrm{curl}\,\mathbf{f}) t\, \mathrm{d}\mathbf{x} + v \int_\Omega (\mathrm{curl}\,\mathbf{u}) t\, \mathrm{d}\mathbf{x}
\end{aligned} \tag{4}$$

where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are given by

$$\begin{gathered}
a(\mathbf{u}, \mathbf{v}) = v \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\, \mathrm{d}\mathbf{x}, \quad b(\mathbf{v}, q) = -\int_\Omega (\mathrm{div}\,\mathbf{v}) q\, \mathrm{d}\mathbf{x}\\
c(z, t) = v \int_\Omega z t\, \mathrm{d}\mathbf{x}
\end{gathered} \tag{5}$$

while the trilinear forms $A(\cdot;\cdot,\cdot)$ and $C(\cdot;\cdot,\cdot)$ associated with the nonlinear terms are defined, in a formal way for the moment, by

$$A(z; \mathbf{u}, \mathbf{v}) = \int_\Omega (z \times \mathbf{u}) \cdot \mathbf{v}\, \mathrm{d}\mathbf{x}, \quad C(\mathbf{u}; z, t) = \alpha \int_\Omega (\mathbf{u} \cdot \nabla z) t\, \mathrm{d}\mathbf{x} \tag{6}$$

It follows from the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ and $L^2(\Omega)$ that system (4) is equivalent to problem (2) when all the equations in this problem are taken in the distribution sense.

The forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are continuous on $H^1(\Omega)^2 \times H^1(\Omega)^2$, $H^1(\Omega)^2 \times L^2(\Omega)$ and $L^2(\Omega) \times L^2(\Omega)$, respectively. Moreover the form $a(\cdot,\cdot)$ is elliptic on $H_0^1(\Omega)^2$. As usual for the Stokes problem, we introduce the subspace

$$V(\Omega)=\{\mathbf{v}\in H_0^1(\Omega)^2;\ \operatorname{div}\mathbf{v}=0\ \text{in}\ \Omega\} \tag{7}$$

Indeed, since the range of $H_0^1(\Omega)^2$ by the divergence operator is contained in $L_0^2(\Omega)$, $V(\Omega)$ coincides with the kernel of $b(\cdot,\cdot)$, hence is closed in $H_0^1(\Omega)^2$ (as follows from the continuity of $b(\cdot,\cdot)$). This leads to the reduced problem:

Find $(\mathbf{u},z)$ in $V(\Omega)\times L^2(\Omega)$ such that

$$\begin{aligned}
&\forall\mathbf{v}\in V(\Omega),\quad a(\mathbf{u},\mathbf{v})+A(z;\mathbf{u},\mathbf{v})=\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\
&\forall t\in L^2(\Omega),\quad c(z,t)+C(\mathbf{u};z,t)=\alpha\int_\Omega(\operatorname{curl}\mathbf{f})t\,\mathrm{d}\mathbf{x}+\nu\int_\Omega(\operatorname{curl}\mathbf{u})t\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{8}$$

It is readily checked that, for any solution $(\mathbf{u},p,z)$ of (4), $(\mathbf{u},z)$ is a solution of (8). Conversely, thanks to the inf–sup condition [20, Chapter I, Corollary 2.4]

$$\forall q\in L_0^2(\Omega),\qquad \sup_{\mathbf{v}\in H_0^1(\Omega)^2}\frac{b(\mathbf{v},q)}{|\mathbf{v}|_{H^1(\Omega)^2}}\geqslant\beta\|q\|_{L^2(\Omega)} \tag{9}$$

(where β is a positive constant), for any solution $(\mathbf{u},z)$ of (8), there exists a unique p in $L_0^2(\Omega)$ such that $(\mathbf{u},p,z)$ is a solution of (4).

We now recall that, for any z in $L^2(\Omega)$, the bilinear form $A(z;\cdot,\cdot)$ is continuous on $L^4(\Omega)^2\times L^4(\Omega)^2$ and that, for any $\mathbf{u}$ in $H^1(\Omega)^2$, the bilinear form $C(\mathbf{u};\cdot,\cdot)$ is continuous on $Z_\mathbf{u}\times L^2(\Omega)$, where $Z_\mathbf{u}$ stands for the space

$$Z_\mathbf{u}=\{t\in L^2(\Omega);\ \mathbf{u}\cdot\nabla t\in L^2(\Omega)\} \tag{10}$$

Moreover, these two forms are antisymmetric, in the sense that

$$\forall z\in L^2(\Omega),\ \forall\mathbf{v}\in L^4(\Omega)^2,\quad A(z;\mathbf{v},\mathbf{v})=0 \tag{11}$$

$$\forall\mathbf{u}\in V(\Omega),\ \forall t\in Z_\mathbf{u},\quad C(\mathbf{u};t,t)=0 \tag{12}$$

(we check (12) thanks to the density of the space $\mathscr{D}(\bar{\Omega})^2$ of infinitely differentiable vector fields in $Z_\mathbf{u}$, which is established in Reference [13, Theorem 3.12]). So taking $\mathbf{v}$ equal to $\mathbf{u}$ and t equal to z in the first and second line of (8), respectively, and using (9) lead to the following statement.

Lemma 2.1

There exists a constant c only depending on the geometry of Ω such that the following *a priori* estimates hold for any solution $(\mathbf{u},p,z)$ of problem (4):

$$\begin{aligned}
\nu|\mathbf{u}|_{H^1(\Omega)^2}\leqslant c\|\mathbf{f}\|_{L^2(\Omega)^2},\quad \nu\|z\|_{L^2(\Omega)}\leqslant c(\|\mathbf{f}\|_{L^2(\Omega)^2}+|\alpha|\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)})\\
\|p\|_{L^2(\Omega)}\leqslant c\|\mathbf{f}\|_{L^2(\Omega)^2}(1+\nu^{-2}(\|\mathbf{f}\|_{L^2(\Omega)^2}+|\alpha|\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)}))
\end{aligned} \tag{13}$$

From these estimates, it can be checked that, for any fixed z in $L^2(\Omega)$, the first two lines in (4) have a unique solution $(\mathbf{u}, p)$ and that, for any fixed $\mathbf{u}$ in $V(\Omega)$, the third line in (4) has a unique solution z (this follows from the analysis of the standard transport equation, see for instance Reference [21] or Reference [22, Lemma 4.4.4.1] for the existence and References [23, 13] for the uniqueness). Establishing the existence of a solution for the global system relies on Brouwer's fixed point theorem, the details of this proof can be found in Reference [13, Section 2].

Theorem 2.2
For any data $\mathbf{f}$ in $H(\mathrm{curl}, \Omega)$, problem (4) admits a solution $(\mathbf{u}, p, z)$ in $H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega)$. Moreover the function z belongs to $Z_{\mathbf{u}}$.

Remark 2.3
A sufficient condition on the data for the uniqueness of the solutions $(\mathbf{u}, p, z)$ of problem (4) such that z belongs to $H^1(\Omega)$ is given in Reference [14], however this condition seems too restrictive in most practical situations.

The regularity properties of the solutions of problem (4) are proven in Reference [14] thanks to arguments in References [15, 22, 24], see also References [25, 26]. Their proof relies on the idea that the first two lines in (4) can be interpreted as a Stokes problem with data $\mathbf{f} - z \times \mathbf{u}$ together with a boot-strapping argument on this problem. We sum up these results in the next two propositions.

Proposition 2.4
For any data $\mathbf{f}$ in $H(\mathrm{curl}, \Omega)$, any solution $(\mathbf{u}, p, z)$ of problem (4) is such that the pair $(\mathbf{u}, p)$ belongs

(i) to $W^{2,4/3}(\Omega)^2 \times W^{1,4/3}(\Omega)$,
(ii) and moreover, if the domain Ω is convex, to $H^2(\Omega)^2 \times H^1(\Omega)$.

Furthermore, if $\mathrm{curl}\,\mathbf{f}$ belongs to $L^p(\Omega)$, the function z belongs to $L^p(\Omega)$, for all $p < +\infty$.

Further regularity on z, namely the fact that z belongs to $H^1(\Omega)$, is established only with a further assumption on $\mathbf{u}$ (note however that $\mathbf{u}$ belongs to $W^{1,\infty}(\Omega)^2$ if Ω is a convex polygon and $\mathrm{curl}\,\mathbf{f}$ belongs to $L^p(\Omega)$ for some $p > 2$). We refer to References [14, 15] for the proof of the following statement in the case of a convex polygon, but it can be checked that, since the function $\mathbf{u}$ vanishes on $\partial\Omega$, this regularity property still holds with the same assumptions for nonconvex domains Ω (this can be checked by extending $\mathbf{u}$ by zero to an open ball containing $\bar{\Omega}$ and writing the transport equation in this ball).

Proposition 2.5
For any data $\mathbf{f}$ in $H^1(\Omega)^2$ and any solution $(\mathbf{u}, p, z)$ of problem (4) such that the velocity $\mathbf{u}$ belongs to $H^2(\Omega)^2 \cap W^{1,\infty}(\Omega)^2$ and satisfies

$$|\alpha| |\mathbf{u}|_{W^{1,\infty}(\Omega)^2} < v \tag{14}$$

the function z belongs to $H^1(\Omega)$.

3. THE REGULARIZED PROBLEM

Let ε be a parameter which satisfies $0<\varepsilon\leqslant 1$. We are interested in the regularized problem

$$\begin{aligned}
&-v\Delta\mathbf{u}_\varepsilon+\mathbf{grad}\,p_\varepsilon+z_\varepsilon\times\mathbf{u}_\varepsilon=\mathbf{f}\quad\text{in }\Omega\\
&\operatorname{div}\mathbf{u}_\varepsilon=0\quad\text{in }\Omega\\
&-\varepsilon\Delta z_\varepsilon+vz_\varepsilon+\alpha\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon=\alpha\operatorname{curl}\mathbf{f}+v\operatorname{curl}\mathbf{u}_\varepsilon\quad\text{in }\Omega\\
&\mathbf{u}_\varepsilon=\mathbf{0}\quad\text{on }\partial\Omega\\
&\varepsilon\partial_n z_\varepsilon=0\quad\text{on }\partial\Omega
\end{aligned}\tag{15}$$

where the equation on z_ε is now of Helmholtz type. The introduction of the Laplace operator requires a boundary condition on z_ε: Here, we have chosen a Neumann condition, as usual in a regularization process. In a first step we write the variational formulation of this problem and prove that it admits a solution. We study the convergence of this solution when ε tends to zero. Next, we prove an estimate for the distance between the solutions of problems (2) and (15).

With the notation of Section 2, the variational formulation of problem (15) can be written

$$\begin{aligned}
&\text{Find }(\mathbf{u}_\varepsilon,p_\varepsilon,z_\varepsilon)\text{ in }H_0^1(\Omega)^2\times L_0^2(\Omega)\times H^1(\Omega)\text{ such that}\\
&\forall\mathbf{v}\in H_0^1(\Omega)^2,\quad a(\mathbf{u}_\varepsilon,\mathbf{v})+b(\mathbf{v},p_\varepsilon)+A(z_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\
&\forall q\in L_0^2(\Omega),\quad b(\mathbf{u}_\varepsilon,q)=0\\
&\forall t\in H^1(\Omega),\quad c_\varepsilon(z_\varepsilon,t)+C(\mathbf{u}_\varepsilon;z_\varepsilon,t)=\alpha\int_\Omega(\operatorname{curl}\mathbf{f})t\,\mathrm{d}\mathbf{x}+v\int_\Omega(\operatorname{curl}\mathbf{u}_\varepsilon)t\,\mathrm{d}\mathbf{x}
\end{aligned}\tag{16}$$

where the bilinear form $c_\varepsilon(\cdot,\cdot)$ is now given by

$$c_\varepsilon(z,t)=\varepsilon\int_\Omega\mathbf{grad}\,z\cdot\mathbf{grad}\,t\,\mathrm{d}\mathbf{x}+v\int_\Omega zt\,\mathrm{d}\mathbf{x}$$

The equivalence of this formulation with problem (15) is readily checked by using density results.

A reduced formulation of problem (16) involves the kernel $V(\Omega)$ defined in (7). It reads:

$$\begin{aligned}
&\text{Find }(\mathbf{u}_\varepsilon,z_\varepsilon)\text{ in }V(\Omega)\times H^1(\Omega)\text{ such that}\\
&\forall\mathbf{v}\in V(\Omega),\quad a(\mathbf{u}_\varepsilon,\mathbf{v})+A(z_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\
&\forall t\in H^1(\Omega),\quad c_\varepsilon(z_\varepsilon,t)+C(\mathbf{u}_\varepsilon;z_\varepsilon,t)=\alpha\int_\Omega(\operatorname{curl}\mathbf{f})t\,\mathrm{d}\mathbf{x}+v\int_\Omega(\operatorname{curl}\mathbf{u}_\varepsilon)t\,\mathrm{d}\mathbf{x}
\end{aligned}\tag{17}$$

Relying on this formulation and using the same arguments as for Lemma 2.1, we can derive *a priori* bounds for the solutions of problem (16). For simplicity, we define the ε-dependent norm

$$\|t\|_\varepsilon = c_\varepsilon(t,t)^{1/2} \tag{18}$$

Lemma 3.1
There exists a constant c depending only on the geometry of Ω such that, for all ε, $0<\varepsilon\leqslant 1$, the following *a priori* estimates hold for any solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ of problem (16):

$$\begin{gathered} \nu|\mathbf{u}_\varepsilon|_{H^1(\Omega)^2} \leqslant c\|\mathbf{f}\|_{L^2(\Omega)^2}, \quad \|z_\varepsilon\|_\varepsilon \leqslant \nu^{-1/2}(c\|\mathbf{f}\|_{L^2(\Omega)^2} + |\alpha|\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}) \\ \|p_\varepsilon\|_{L^2(\Omega)} \leqslant c\|\mathbf{f}\|_{L^2(\Omega)^2}(1+\nu^{-2}(\|\mathbf{f}\|_{L^2(\Omega)^2} + |\alpha|\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)})) \end{gathered} \tag{19}$$

In order to prove the existence of a solution of problem (16) or (17), we denote by $\mathscr{X}$ the product space $V(\Omega)\times H^1(\Omega)$ and we consider the mapping Φ_ε defined from $\mathscr{X}$ into its dual space $\mathscr{X}'$ by

$\forall U=(\mathbf{u},z)\in\mathscr{X}, \quad \forall V=(\mathbf{v},t)\in\mathscr{X},$

$$\langle\Phi_\varepsilon(U),V\rangle = a(\mathbf{u},\mathbf{v}) + A(z;\mathbf{u},\mathbf{v}) - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + c_\varepsilon(z,t) + C(\mathbf{u};z,t) - \alpha\int_\Omega(\mathrm{curl}\,\mathbf{f})t\,\mathrm{d}\mathbf{x} - \nu\int_\Omega(\mathrm{curl}\,\mathbf{u})t\,\mathrm{d}\mathbf{x}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathscr{X}'$ and $\mathscr{X}$. It can be checked that the mapping Φ_ε is continuous on $\mathscr{X}$ and moreover satisfies for all $U=(\mathbf{u},z)$ in $\mathscr{X}$,

$$\langle\Phi_\varepsilon(U),U\rangle \geqslant \nu|\mathbf{u}|^2_{H^1(\Omega)^2} - c_0\|\mathbf{f}\|_{L^2(\Omega)^2}|\mathbf{u}|_{H^1(\Omega)^2} + \varepsilon|z|^2_{H^1(\Omega)} + \nu\|z\|^2_{L^2(\Omega)} - |\alpha|\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}\|z\|_{L^2(\Omega)} - \nu|\mathbf{u}|_{H^1(\Omega)^2}\|z\|_{L^2(\Omega)}$$

where c_0 denotes the constant of the Poincaré–Friedrichs inequality (note that, since $\mathbf{u}$ belongs to $V(\Omega)$, the inequality $\|\mathrm{curl}\,\mathbf{u}\|_{L^2(\Omega)}\leqslant|\mathbf{u}|_{H^1(\Omega)^2}$ is derived thanks to an integration by parts). This yields

$$\langle\Phi_\varepsilon(U),U\rangle \geqslant \frac{\nu}{4}|\mathbf{u}|^2_{H^1(\Omega)^2} - \frac{c_0^2}{\nu}\|\mathbf{f}\|^2_{L^2(\Omega)^2} + \varepsilon|z|^2_{H^1(\Omega)} + \frac{\nu}{4}\|z\|^2_{L^2(\Omega)} - \frac{\alpha^2}{\nu}\|\mathrm{curl}\,\mathbf{f}\|^2_{L^2(\Omega)} \tag{20}$$

The existence result relies on this last estimate.

Theorem 3.2
For any data $\mathbf{f}$ in $H(\mathrm{curl},\Omega)$, problem (16) admits a solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ in $H^1_0(\Omega)^2\times L^2_0(\Omega)\times H^1(\Omega)$.

Proof
It is performed in three steps.

(1) Let $(\mathscr{X}_n)_n$ be an increasing sequence of finite-dimensional subspaces of $\mathscr{X}$ such that their union is dense in $\mathscr{X}$ (the existence of such a sequence is due to the separability

of $\mathscr{X}$). We set:

$$\mu^2 = \frac{c_0^2}{\nu}\|\mathbf{f}\|^2_{L^2(\Omega)^2} + \frac{\alpha^2}{\nu}\|\operatorname{curl}\mathbf{f}\|^2_{L^2(\Omega)}$$

For all n, using (20), we see that $\langle \Phi_\varepsilon(U_n), U_n\rangle$ is nonnegative for all $U_n = (\mathbf{u}_n, z_n)$ in $\mathscr{X}_n$ such that

$$\frac{\nu}{4}|\mathbf{u}_n|^2_{H^1(\Omega)^2} + \varepsilon|z_n|^2_{H^1(\Omega)} + \frac{\nu}{4}\|z_n\|^2_{L^2(\Omega)} = \mu^2$$

So Brouwer's fixed point theorem implies the existence of a solution U_n in $\mathscr{X}_n$ of the problem

$$\forall V_n \in \mathscr{X}_n, \quad \langle \Phi_\varepsilon(U_n), V_n\rangle = 0$$

and such that

$$\frac{\nu}{4}|\mathbf{u}_n|^2_{H^1(\Omega)^2} + \varepsilon|z_n|^2_{H^1(\Omega)} + \frac{\nu}{4}\|z_n\|^2_{L^2(\Omega)} \leqslant \mu^2$$

(2) From the previous estimate, there exists a subsequence $(U_{n'})_{n'}$ of $(U_n)_n$ which converges weakly in $\mathscr{X}$. Since the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$ is compact, there exists another subsequence $(U_{n''})_{n''}$ of $(U_{n'})_{n'}$ which converges towards U_ε strongly in $L^4(\Omega)^2 \times L^4(\Omega)$. Then, standard arguments yield that U_ε is a solution of problem (17).

(3) By setting $U_\varepsilon = (\mathbf{u}_\varepsilon, z_\varepsilon)$ and using the inf–sup condition (9), we derive the existence of a function p_ε in $L^2_0(\Omega)$ such that the triple $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ is a solution of problem (16).

In contrast to problem (4), the uniqueness of the solution of problem (16) does not require any further regularity, however the condition needed for that is rather disappointing. We skip the proof that relies on completely standard arguments, combined with the fact that the norm of the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ behaves like $p^{1/2}$ for large values of p, see Reference [27].

Proposition 3.3
For any data $\mathbf{f}$ in $H(\operatorname{curl},\Omega)$ and any ε such that, for some real number p, $2 < p < +\infty$, and an appropriate constant c only depending on the geometry of Ω,

$$\frac{c}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)^2}\left(1 + \frac{|\alpha| p^{1/2}}{\varepsilon^{1/2}\nu^2}\left(1 + \frac{\nu^{1/p}}{\varepsilon^{1/p}}\right)(\|\mathbf{f}\|_{L^2(\Omega)^2} + |\alpha|\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)})\right) < 1 \tag{21}$$

problem (16) admits at most one solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ in $H^1_0(\Omega)^2 \times L^2_0(\Omega) \times H^1(\Omega)$.

Note that condition (21) for $\alpha = 0$ is exactly the sufficient condition for the uniqueness of the solution of Navier–Stokes equations, see Reference [20, Chapter IV, Theorem 2.2]. The dependence on ε in condition (21) can be optimized through an appropriate choice of p. However, even for this choice, this condition is not satisfied in practical situations, so it is not used in what follows. The regularity properties of any solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ are now easily derived from the regularity properties of the Stokes problem and the Laplace equation together with a boot-strapping argument, see Reference [22] for the details of the proof.

Proposition 3.4
For any data $\mathbf{f}$ in $H(\mathrm{curl},\Omega)$, any solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ of problem (16) belongs to $H^{s+1}(\Omega)^2 \times H^s(\Omega) \times H^{s+1}(\Omega)$,

(i) for all s, $0 \leqslant s \leqslant \frac{1}{2}$, in the general case,
(ii) for all $s \leqslant s_0$ when Ω is a polygon, where the parameter s_0 satisfies $\frac{1}{2} < s_0 \leqslant 1$ and only depends on the largest inner angle of Ω,
(iii) for all $s \leqslant 1$ when Ω is convex.

Moreover, for these values of s, there exists a constant $c_{\mathbf{f}}$, only depending on the geometry of Ω, the norm of $\mathbf{f}$ in $H(\mathrm{curl},\Omega)$, ν, α and s, such that

$$\|\mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2} + \|p_\varepsilon\|_{H^s(\Omega)} + \varepsilon^{s+1/2}\|z_\varepsilon\|_{H^{s+1}(\Omega)} \leqslant c_{\mathbf{f}} \tag{22}$$

We now prove the convergence of a subsequence of the solutions $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ of problem (16) when ε tends to zero. From now on, we assume that the data $\mathbf{f}$ belong to $H(\mathrm{curl},\Omega)$.

Theorem 3.5
There exists a sequence $(\mathbf{u}_n, p_n, z_n)_n$ of $H_0^1(\Omega)^2 \times L_0^2(\Omega) \times H^1(\Omega)$ such that

(i) each $(\mathbf{u}_n, p_n, z_n)_n$ is a solution of problem (16) with $\varepsilon = \varepsilon_n$,
(ii) the sequence $(\varepsilon_n)_n$ tends to zero,
(iii) the sequence $(\mathbf{u}_n, p_n, z_n)_n$ converges towards a solution $(\mathbf{u}, p, z)$ of problem (4) weakly in $H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega)$.

Proof
It follows from Lemma 3.1 that the family of solutions $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ is bounded in $H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega)$ independently of ε. Moreover, it follows from the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$ that each product $z_\varepsilon \times \mathbf{u}_\varepsilon$ is bounded in $L^{4/3}(\Omega)^2$. Thus writing the first line of problem (16) as

$$\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad a(\mathbf{u}_\varepsilon, \mathbf{v}) + b(\mathbf{v}, \mathbf{p}_\varepsilon) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} - A(z_\varepsilon; \mathbf{u}_\varepsilon, \mathbf{v})$$

and using the standard regularity properties of the Stokes problem [22] yield that each $\mathbf{u}_\varepsilon$ belongs to $W^{2,4/3}(\Omega)^2$ and satisfies

$$\|\mathbf{u}_\varepsilon\|_{W^{2,4/3}(\Omega)^2} \leqslant c\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)^2}(1 + \nu^{-2}(\|\mathbf{f}\|_{L^2(\Omega)^2} + |\alpha|\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}) \tag{23}$$

By combining (19) and (23), we derive the existence of a sequence $(\mathbf{u}_n, p_n, z_n)_n$ satisfying parts (i) and (ii) of the theorem, which converges towards a triple $(\mathbf{u}, z, p)$ weakly in $(H^{3/2}(\Omega) \cap H_0^1(\Omega))^2 \times L_0^2(\Omega) \times L^2(\Omega)$ and such that the sequence $(\mathbf{u}_n)_n$ converges towards $\mathbf{u}$ strongly in $H^{4/3}(\Omega)^2$ for instance. So it remains to check that $(\mathbf{u}, p, z)$ is a solution of problem (4).

(1) In the first line of problem (16), since $(\mathbf{u}_n)_n$ converges strongly in $L^\infty(\Omega)^2$, the sequence of nonlinear terms $(z_n \times \mathbf{u}_n)$ converges to $z \times \mathbf{u}$ weakly in $L^2(\Omega)$. So the triple $(\mathbf{u}, p, z)$ satisfies the first line of (4).

(2) Since the second line of (16) is a linear equation, it is obvious that the triple $(\mathbf{u}, p, z)$ satisfies the second line of (4).
(3) In the third line of (16), we consider a fixed t in $H^1(\Omega)$ and we note that

$$\left|\varepsilon_n \int_\Omega \mathbf{grad}\, z_n \cdot \mathbf{grad}\, t\, \mathrm{d}\mathbf{x}\right| \leqslant \varepsilon_n^{1/2} \|z_n\|_{\varepsilon_n} |t|_{H^1(\Omega)}$$

So it follows from (19) that this term tends to zero. On the other hand, we derive by integration by parts that

$$C(\mathbf{u}_n; z_n, t) = -C(\mathbf{u}_n; t, z_n)$$

Then, the strong convergence of $(\mathbf{u}_n)_n$ in $L^\infty(\Omega)^2$ implies that

$$\lim_{n\to+\infty} C(\mathbf{u}_n; z_n, t) = -C(\mathbf{u}; t, z)$$

Letting now t run through $\mathscr{D}(\Omega)$ yields that the pair $(\mathbf{u}, z)$ satisfies the third line of problem (2) in the distribution sense. Since z, $\operatorname{curl}\mathbf{u}$ and $\operatorname{curl}\mathbf{f}$ belong to $L^2(\Omega)$, this implies that the function z belongs to the space $Z_\mathbf{u}$ defined in (10). So, by combining all this with a density argument, we deduce that the triple $(\mathbf{u}, p, z)$ satisfies the third line of (4) for all t in $L^2(\Omega)$.
This concludes the proof.

By combining the previous arguments with an energy inequality (see Reference [13] for analogous results), we can prove that there exists a subsequence of the previous sequence $(\mathbf{u}_n, p_n, z_n)_n$ which converges towards a solution $(\mathbf{u}, p, z)$ of problem (4) strongly in $H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega)$ (or even strongly in $H^{s+1}(\Omega)^2 \times H^s(\Omega) \times L^2(\Omega)$, $0 < s < \frac{1}{2}$). We conclude with a first *a posteriori* type estimate.

In order to prove this estimate, we introduce the Stokes operator $\mathscr{S}$ defined as follows: For any data $\mathbf{f}$ in $H^{-1}(\Omega)^2$, $\mathscr{S}\mathbf{f}$ is equal to the part $\mathbf{u}$ of the solution $(\mathbf{u}, p)$ in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$ of the equations (we do not make explicit the definition of the duality pairings when obvious)

$$\begin{aligned} &\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle \\ &\forall q \in L_0^2(\Omega), \qquad b(\mathbf{u}, q) = 0 \end{aligned} \tag{24}$$

Let also s be a fixed real number, $0 < s < \frac{1}{2}$. It can be noted from the standard regularity properties of the Stokes problem that the operator $\mathscr{S}$ maps $H^{s-1}(\Omega)^2$ into $H^{s+1}(\Omega)^2$. Next, we consider the mapping Ψ which associates with any $\mathbf{u}$ in $H^{s+1}(\Omega)^2$ the solution $z = \Psi(\mathbf{u})$ in $L^2(\Omega)$ of the problem

$$\nu z + \alpha \mathbf{u} \cdot \nabla z = \alpha \operatorname{curl}\mathbf{f} + \nu \operatorname{curl}\mathbf{u} \quad \text{in } \Omega \tag{25}$$

Indeed it follows from Reference [13, Section 3] that this equation has a unique solution z in $L^2(\Omega)$, which moreover belongs to the space $Z_\mathbf{u}$ introduced in (10). Now, it can be observed that a triple $(\mathbf{u}, p, z)$ which belongs to $H^{s+1}(\Omega)^2 \times H^s(\Omega) \times L^2(\Omega)$ is a solution of problem (4) if and only if the velocity $\mathbf{u}$ satisfies

$$\Lambda(\mathbf{u}) = \mathbf{u} + \mathscr{S}(\Psi(\mathbf{u}) \times \mathbf{u} - \mathbf{f}) = \mathbf{0} \tag{26}$$

In a preliminary step, we prove some properties of the mapping Ψ. This requires the introduction of the segment

$$\mathscr{I}_\varepsilon(\mathbf{u}) = \{\mathbf{u} - \tau(\mathbf{u} - \mathbf{u}_\varepsilon),\ 0 \leqslant \tau \leqslant 1\} \tag{27}$$

where both functions $\mathbf{u}$ and $\mathbf{u}_\varepsilon$ are assumed to belong to $V(\Omega) \cap L^\infty(\Omega)^2$.

Lemma 3.6
If $\Psi(\mathscr{I}_\varepsilon(\mathbf{u}))$ is contained in $H^1(\Omega)$, then the mapping Ψ is local Lipschitz-continuous at each function on $\mathscr{I}_\varepsilon(\mathbf{u})$ with values in $L^2(\Omega)$.

Proof
For any functions $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}^*$ in $\mathscr{I}_\varepsilon(\mathbf{u})$, we set $\tilde{z} = \Psi(\tilde{\mathbf{u}})$ and $\tilde{z}^* = \Psi(\tilde{\mathbf{u}}^*)$. From the definition of Ψ, we have for all t in $L^2(\Omega)$,

$$c(\tilde{z}^* - \tilde{z}, t) + C(\tilde{\mathbf{u}}^*; \tilde{z}^*, t) - C(\tilde{\mathbf{u}}; \tilde{z}, t) = v \int_\Omega \mathrm{curl}(\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}) t \, \mathrm{d}\mathbf{x}$$

or equivalently

$$c(\tilde{z}^* - \tilde{z}, t) = v \int_\Omega \mathrm{curl}(\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}) t \, \mathrm{d}\mathbf{x} - C(\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}; \tilde{z}, t) - C(\tilde{\mathbf{u}}^*; \tilde{z}^* - \tilde{z}, t)$$

Taking t equal to $\tilde{z}^* - \tilde{z}$ yields

$$\|\tilde{z}^* - \tilde{z}\|^2_{L^2(\Omega)} \leqslant c(|\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}|_{H^1(\Omega)^2} + \|\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}\|_{L^\infty(\Omega)^2} |\tilde{z}|_{H^1(\Omega)}) \|\tilde{z}^* - \tilde{z}\|_{L^2(\Omega)}$$

whence we derive

$$\|\tilde{z}^* - \tilde{z}\|_{L^2(\Omega)} \leqslant c(1 + |\tilde{z}|_{H^1(\Omega)}) \|\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}\|_{H^1(\Omega)^2 \cap L^\infty(\Omega)^2}$$

So the mapping Ψ is Lipschitz-continuous in $\tilde{\mathbf{u}}$ with a Lipschitz constant only depending on $\|\Psi(\tilde{\mathbf{u}})\|_{H^1(\Omega)}$.

Lemma 3.7
If $\Psi(\mathscr{I}_\varepsilon(\mathbf{u}))$ is contained in a bounded set of $H^1(\Omega)$, then the mapping Ψ is differentiable in the direction $\mathbf{u} - \mathbf{u}_\varepsilon$ at each point of $\mathscr{I}_\varepsilon(\mathbf{u})$. Moreover, if the following assumptions hold:

(i) The velocities $\mathbf{u}$ and $\mathbf{u}_\varepsilon$ belong to $H^2(\Omega)^2 \cap W^{1,\infty}(\Omega)^2$,
(ii) The mapping $D\Psi(\mathbf{u}).(\mathbf{u} - \mathbf{u}_\varepsilon)$ satisfies for a positive constant c_0

$$\|D\Psi(\mathbf{u}).(\mathbf{u} - \mathbf{u}_\varepsilon)\|_{H^1(\Omega)} \leqslant c_0 \|\mathbf{u} - \mathbf{u}_\varepsilon\|_{H^2(\Omega)^2 \cap W^{1,\infty}(\Omega)^2} \tag{28}$$

(iii) The mapping Ψ is Lipschitz-continuous in $\mathbf{u}$ with values in $H^1(\Omega)$,

then the mapping: $\tilde{\mathbf{u}} \mapsto D\Psi(\tilde{\mathbf{u}}).(\mathbf{u} - \mathbf{u}_\varepsilon)$ is Lipschitz-continuous on $\mathscr{I}_\varepsilon(\mathbf{u})$, more precisely it satisfies, for any $\tilde{\mathbf{u}}$ in $\mathscr{I}_\varepsilon(\mathbf{u})$,

$$\|D\Psi(\tilde{\mathbf{u}}).(\mathbf{u} - \mathbf{u}_\varepsilon) - D\Psi(\mathbf{u}).(\mathbf{u} - \mathbf{u}_\varepsilon)\|_{L^2(\Omega)} \leqslant c \|\tilde{\mathbf{u}} - \mathbf{u}\|_{H^1(\Omega)^2 \cap L^\infty(\Omega)^2} \|\mathbf{u} - \mathbf{u}_\varepsilon\|_{H^2(\Omega)^2 \cap W^{1,\infty}(\Omega)^2} \tag{29}$$

Proof
We check separately the two assertions.

(1) For any function $\tilde{\mathbf{u}}$ in $\mathscr{I}_\varepsilon(\mathbf{u})$, we now set $\tilde{\mathbf{u}}^* = \tilde{\mathbf{u}} + \tau(\mathbf{u} - \mathbf{u}_\varepsilon)$, $\tilde{z} = \Psi(\tilde{\mathbf{u}})$ and $\tilde{z}^* = \Psi(\tilde{\mathbf{u}}^*)$, for any $\tau, -1 < \tau < 1$, such that $\tilde{\mathbf{u}}^*$ belongs to $\mathscr{I}_\varepsilon(\mathbf{u})$. As in the previous proof, we observe that, for all t in $L^2(\Omega)$,

$$c(\tilde{z}^* - \tilde{z}, t) + C(\tilde{\mathbf{u}}^*; \tilde{z}^*, t) - C(\tilde{\mathbf{u}}; \tilde{z}, t) = v\tau \int_\Omega \operatorname{curl}(\mathbf{u} - \mathbf{u}_\varepsilon) t \, d\mathbf{x}$$

or equivalently that

$$c\left(\frac{\tilde{z}^* - \tilde{z}}{\tau}, t\right) + C\left(\tilde{\mathbf{u}}; \frac{\tilde{z}^* - \tilde{z}}{\tau}, t\right) + C(\mathbf{u} - \mathbf{u}_\varepsilon; \tilde{z}^*, t) = v \int_\Omega \operatorname{curl}(\mathbf{u} - \mathbf{u}_\varepsilon) t \, d\mathbf{x}$$

Thus taking t equal to $(\tilde{z}^* - \tilde{z})/\tau$ and noting that $\tilde{z}^*$ belongs to a bounded set of $H^1(\Omega)$ yield that the quantities, $\zeta_\tau = (\tilde{z}^* - \tilde{z})/\tau$ are bounded independently of τ in $L^2(\Omega)$. So there exists a sequence $(\tau_n)_n$ tending to zero such that $(\zeta_{\tau_n})_n$ converges to a function $\tilde{\zeta}$ weakly in $L^2(\Omega)$. We also derive that, for any t in $\mathscr{D}(\Omega)$,

$$\lim_{n \to +\infty} C(\tilde{\mathbf{u}}; \zeta_{\tau_n}, t) = -\lim_{n \to +\infty} C(\tilde{\mathbf{u}}; t, \zeta_{\tau_n}) = -C(\tilde{\mathbf{u}}; t, \tilde{\zeta}) = \alpha \langle \tilde{\mathbf{u}} \cdot \nabla \tilde{\zeta}, t \rangle$$

the last product being taken in the distributional sense. Combining all this implies that $\tilde{\zeta} = D\Psi(\tilde{\mathbf{u}}).(\mathbf{u} - \mathbf{u}_\varepsilon)$ is independent of the sequence $(\tau_n)_n$, belongs to the space $Z_{\tilde{\mathbf{u}}}$ and is the unique solution of the equation

$$\forall t \in L^2(\Omega), \quad c(\tilde{\zeta}, t) + C(\tilde{\mathbf{u}}; \tilde{\zeta}, t) + C(\mathbf{u} - \mathbf{u}_\varepsilon; \tilde{z}, t) = v \int_\Omega \operatorname{curl}(\mathbf{u} - \mathbf{u}_\varepsilon) t \, d\mathbf{x} \tag{30}$$

(2) Let ζ denote the solution of (30) for $\tilde{\mathbf{u}} = \mathbf{u}$ (and $\tilde{z} = \Psi(\mathbf{u})$). We have, for all t in $L^2(\Omega)$,

$$c(\tilde{\zeta} - \zeta, t) = -C(\tilde{\mathbf{u}} - \mathbf{u}; \zeta, t) - C(\tilde{\mathbf{u}}; \tilde{\zeta} - \zeta, t) - C(\mathbf{u} - \mathbf{u}_\varepsilon; \tilde{z} - z, t)$$

Thus, taking t equal to $\tilde{\zeta} - \zeta$ and using the following estimates:

$$|C(\tilde{\mathbf{u}} - \mathbf{u}; \zeta, \tilde{\zeta} - \zeta)| \leqslant |\alpha| \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(\Omega)^2} |\zeta|_{H^1(\Omega)} \|\tilde{\zeta} - \zeta\|_{L^2(\Omega)}$$

$$|C(\mathbf{u} - \mathbf{u}_\varepsilon; \tilde{z} - z, \tilde{\zeta} - \zeta)| \leqslant |\alpha| \|\mathbf{u} - \mathbf{u}_\varepsilon\|_{L^\infty(\Omega)^2} |\tilde{z} - z|_{H^1(\Omega)} \|\tilde{\zeta} - \zeta\|_{L^2(\Omega)}$$

yield that

$$\|\tilde{\zeta} - \zeta\|_{L^2(\Omega)} \leqslant c(\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(\Omega)^2} |\zeta|_{H^1(\Omega)} + \|\mathbf{u} - \mathbf{u}_\varepsilon\|_{L^\infty(\Omega)^2} |\tilde{z} - z|_{H^1(\Omega)})$$

Using assumptions (ii) and (iii) to bound the two terms in the right-hand side leads to the desired property.

Remark 3.8
The assumptions of Lemmas 3.6 and 3.7 hold in the case $\alpha = 0$ of the Navier–Stokes equations whenever Ω is convex. They require some further regularity of the solution in the case $\alpha \neq 0$

(for instance the derivative of order 2 of $\Psi(\mathbf{u})$ in the direction $\mathbf{u}-\mathbf{u}_\varepsilon$ must belong to $L^2(\Omega)$). However they seem less restrictive than condition (14).

The same arguments as for the proof of Lemma 3.7 imply that the function Ψ is Gâteaux–differentiable at $\mathbf{u}$ on $H^s(\Omega)^2$, $s>1$. Indeed, for each $\mathbf{w}$ in $H^s(\Omega)^2$, the function $\zeta = D\Psi(\mathbf{u}).\mathbf{w}$ is the unique solution in $L^2(\Omega)$ of the problem

$$\zeta + \mathbf{u}\cdot\nabla\zeta + \mathbf{w}\cdot\nabla\Psi(\mathbf{u}) = v\,\mathrm{curl}\,\mathbf{w} \quad \text{in } \Omega \tag{31}$$

So we are in a position to state the next proposition. The idea for its proof is due to Pousin and Rappaz, see Reference [28, Theorem 3] and also Reference [16, Proposition 2.1] for a modified version. However the assumptions which are made in both references are not satisfied in the present situation, so we prove the desired result directly.

Proposition 3.9

Let $U=(\mathbf{u},z)$ be a solution of problem (8) in $(H^2(\Omega)^2\cap W^{1,\infty}(\Omega)^2)\times H^1(\Omega)$ such that the operator

$$D\Lambda(\mathbf{u}) = \mathscr{I} + \mathscr{S}D(\Psi(\mathbf{u})\times\mathbf{u}) \tag{32}$$

is an automorphism of $H^{s+1}(\Omega)^2$ for a real number s, $0<s<\frac{1}{2}$, and that the mapping Ψ is Lipschitz-continuous in $\mathbf{u}$ with values in $H^1(\Omega)$. There exist a neighbourhood $\mathscr{U}$ of U in $(H^2(\Omega)^2\cap W^{1,\infty}(\Omega)^2)\times H^1(\Omega)$ and a constant $c(U)$ only depending on U and s such that any solution $U_\varepsilon=(\mathbf{u}_\varepsilon,z_\varepsilon)$ of problem (17) in $\mathscr{U}$ such that

(i) $\Psi(\mathscr{I}_\varepsilon(\mathbf{u}))$ is contained in a bounded set of $H^1(\Omega)$,
(ii) inequality (28) holds for the mapping $D\Psi(\mathbf{u}).(\mathbf{u}-\mathbf{u}_\varepsilon)$,

satisfies

$$\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2} \leqslant c(U)\varepsilon\|\Delta z_\varepsilon\|_{L^2(\Omega)} \tag{33}$$

Proof

We define the function λ from [0,1] into $H^{s+1}(\Omega)^2$ by

$$\lambda(\tau) = \Lambda(\mathbf{u}-\tau(\mathbf{u}-\mathbf{u}_\varepsilon)), \quad 0\leqslant\tau\leqslant 1$$

It follows from Lemma 3.7 that this function is continuously differentiable on [0,1]. So we have

$$\Lambda(\mathbf{u}_\varepsilon) = \Lambda(\mathbf{u}_\varepsilon)-\Lambda(\mathbf{u}) = \lambda(1)-\lambda(0) = \int_0^1 \lambda'(\tau)\,\mathrm{d}\tau = \lambda'(0) + \int_0^1 (\lambda'(\tau)-\lambda'(0))\,\mathrm{d}\tau$$

We note that $\lambda'(0)$ is equal to $-D\Lambda(\mathbf{u}).(\mathbf{u}-\mathbf{u}_\varepsilon)$ and observe from Lemmas 3.6 and 3.7 that λ' is Lipschitz-continuous in 0. So denoting by γ the norm of the inverse of $D\Lambda(\mathbf{u})$ and choosing $\mathscr{U}$ such that (see (29))

$$\|\lambda'(\tau)-\lambda'(0)\|_{H^{s+1}(\Omega)^2} \leqslant \frac{\tau}{2\gamma}\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2}$$

yield the existence of a constant $c_0(U)$ such that

$$\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2} \leqslant c_0'(U)\|\Lambda(\mathbf{u}_\varepsilon)\|_{H^{s+1}(\Omega)^2}$$

In order to estimate the residual $\|\Lambda(\mathbf{u}_\varepsilon)\|_{H^{s+1}(\Omega)^2}$, we note that

$$\Lambda(\mathbf{u}_\varepsilon) = -\mathscr{S}((\Psi_\varepsilon - \Psi)(\mathbf{u}_\varepsilon) \times \mathbf{u}_\varepsilon)$$

where $\Psi_\varepsilon(\mathbf{u}_\varepsilon)$ denotes the part z_ε of the solution U_ε. Setting $\tilde{z}_\varepsilon = \Psi(\mathbf{u}_\varepsilon)$, we have

$$v(z_\varepsilon - \tilde{z}_\varepsilon) + \alpha \mathbf{u}_\varepsilon \cdot \nabla(z_\varepsilon - \tilde{z}_\varepsilon) = \varepsilon \Delta z_\varepsilon$$

Multiplying this equation by $z_\varepsilon - \tilde{z}_\varepsilon$ yields that

$$\|z_\varepsilon - \tilde{z}_\varepsilon\|_{L^2(\Omega)} \leqslant \tfrac{\varepsilon}{v} \|\Delta z_\varepsilon\|_{L^2(\Omega)}$$

and combining this with (19) and the standard properties of $\mathscr{S}$ gives estimate (33).

Of course, an estimate of the errors $p - p_\varepsilon$ and $z - z_\varepsilon$ can be derived from (33) in a usual way.

Corollary 3.10
If the assumption of Proposition 3.9 holds, there exists a constant $c'(U)$ only depending on U such that any solution $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ of problem (16) with $(\mathbf{u}_\varepsilon, z_\varepsilon)$ in $\mathscr{U}$ satisfies

$$\|p - p_\varepsilon\|_{L^2(\Omega)} + \|z - z_\varepsilon\|_{L^2(\Omega)} \leqslant c'(U)\varepsilon \|\Delta z_\varepsilon\|_{L^2(\Omega)} \tag{34}$$

Proof
We first prove the estimate for $\|z - z_\varepsilon\|_{L^2(\Omega)}$. Indeed, we have

$$vz + \alpha \mathbf{u} \cdot \nabla z = \alpha \operatorname{curl} \mathbf{f} + v \operatorname{curl} \mathbf{u}, \quad vz_\varepsilon + \alpha \mathbf{u}_\varepsilon \cdot \nabla z_\varepsilon = \alpha \operatorname{curl} \mathbf{f} + v \operatorname{curl} \mathbf{u}_\varepsilon + \varepsilon \Delta z_\varepsilon$$

Subtracting the second equation from the first one, multiplying by $z - z_\varepsilon$ and using the same arguments as in the proof of Proposition 3.6 give

$$v\|z - z_\varepsilon\|_{L^2(\Omega)} \leqslant c(1 + \|z\|_{H^1(\Omega)}) \|\mathbf{u} - \mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2} + \varepsilon \|\Delta z_\varepsilon\|_{L^2(\Omega)}$$

so that the desired estimate follows from (33). The estimate for $p - p_\varepsilon$ is then derived from this estimate and (33), combined with the inf–sup condition (9).

Remark 3.11
Note that, in the assumption of Proposition 3.9 and in all the previous estimates, the space $H^{s+1}(\Omega)$ can be replaced by $H^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3.12
It is readily checked that, if $(\mathbf{u}_n, p_n, z_n)_n$ denotes the sequence exhibited in Theorem 3.5, then

$$\liminf_{n \to +\infty} \varepsilon_n \|\Delta z_n\|_{L^2(\Omega)} = 0 \tag{35}$$

Note to conclude that the main assumption of Proposition 3.9, i.e. the fact that $D\Lambda(\mathbf{u})$ is an automorphism of $H^{s+1}(\Omega)^2$, can equivalently be expressed as follows: For any data $\mathbf{g}$

in $H^{s-1}(\Omega)^2$, the following problem

$$\begin{aligned} &\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad a(\mathbf{w},\mathbf{v}) + b(\mathbf{v},\pi) + A(z;\mathbf{w},\mathbf{v}) + A(r;\mathbf{u},\mathbf{v}) = \langle \mathbf{g},\mathbf{v}\rangle \\ &\forall q \in L_0^2(\Omega), \quad b(\mathbf{w},q) = 0 \\ &\forall t \in L^2(\Omega), \quad c(r,t) + C(\mathbf{u};r,t) + C(\mathbf{w};z,t) = v\int_\Omega (\operatorname{curl}\mathbf{w})t\,\mathrm{d}\mathbf{x} \end{aligned}$$

has a unique solution $(\mathbf{w},\pi,r)$ in $(H_0^1(\Omega)\cap H^{s+1}(\Omega))^2 \times L_0^2(\Omega)\times L^2(\Omega)$
This assumption seems less restrictive than the uniqueness condition, see Remark 2.3.

4. THE CONFORMING DISCRETE PROBLEM

From now on, we work with a fixed, positive value of ε, and we describe a conforming finite element discretization of problem (16) which is obtained by the Galerkin method. For simplicity we assume that the domain Ω is a polygon. We introduce a regular family $(\mathcal{T}_h)_h$ of triangulations by closed triangles, in the usual sense that

- for each $h, \bar{\Omega}$ is the union of all elements of $\mathcal{T}_h$,
- for each h, the intersection of two different elements of $\mathcal{T}_h$, if not empty, is a corner or a whole edge of both of them,
- the ratio of the diameter h_K of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle is bounded by a constant σ independent of K and h.

As standard, h denotes the maximum of the diameters of the elements of $\mathcal{T}_h$. For any K in $\mathcal{T}_h$, we denote by $\mathbf{n}_K$ the unit outward normal vector to K on ∂K. In all that follows, c stands for a generic constant independent of h and ε but possibly depending on the parameters α and v.

In order to describe the discretization, we first present the conforming approximation of the Stokes operator $\mathcal{S}$ defined by problem (24). The discrete space of pressures is defined by

$$\mathbb{M}_h = \{q_h \in L_0^2(\Omega);\ \forall K \in \mathcal{T}_h,\ q_{h|K} \in \mathcal{P}_0(K)\} \tag{36}$$

where $\mathcal{P}_0(K)$ denotes the space of constant functions on K. As far as the discrete space of velocities is concerned, two choices are considered:

$$\mathbb{X}_h = \{\mathbf{v}_h \in H_0^1(\Omega)^2;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_{h|K} \in \mathcal{P}(K)\} \tag{37}$$

where $\mathcal{P}(K)$ is either

- the space $\mathcal{P}_2(K)^2$ of restrictions to K of polynomials with degree $\leqslant 2$,
- the space spanned by the subspace $\mathcal{P}_1(K)^2$ of affine vector fields on K and the three functions ψ_e associated with the three edges e of K, equal to the normal vector $\mathbf{n}_K$ on e times the product of the two barycentric coordinates associated with the endpoints of e (the corresponding finite element is studied in Reference [29]).

Next, we introduce the discrete Stokes operator $\mathcal{S}_h$ which, with any data $\mathbf{f}$ in $H^{-1}(\Omega)^2$, associates the velocity $\mathbf{u}_h$ of the solution $(\mathbf{u}_h, p_h)$ in $\mathbb{X}_h \times \mathbb{M}_h$ of the system

$$\begin{aligned} &\forall \mathbf{v}_h \in \mathbb{X}_h, \quad a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) = \langle \mathbf{f},\mathbf{v}_h\rangle \\ &\forall q_h \in \mathbb{M}_h, \quad b(\mathbf{u}_h,q_h) = 0 \end{aligned} \tag{38}$$

Indeed, the following inf–sup condition is proven in Reference [20, Chapter II, Section 2] (see also Reference [29, Lemma II.4]): There exists a positive constant β independent of h such that

$$\forall q_h \in \mathbb{M}_h, \quad \sup_{\mathbf{v}_h \in \mathbb{X}_h} \frac{b(\mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{H^1(\Omega)^2}} \geqslant \beta \|q_h\|_{L^2(\Omega)} \tag{39}$$

so that problem (38) has a unique solution.

We also recall from Reference [20, Chapter II, Section 2] two results concerning the operator $\mathscr{S}_h$:

(i) The following stability property holds

$$\|\mathscr{S}_h \mathbf{f}\|_{H^1(\Omega)^2} \leqslant c\|\mathbf{f}\|_{H^{-1}(\Omega)^2} \tag{40}$$

(ii) If the data $\mathbf{f}$ belongs to $H^{s-1}(\Omega)^2$ and the solution $\mathscr{S}\mathbf{f}$ belongs to $H^{s+1}(\Omega)^2$, $0<s\leqslant 1$, the following error estimate holds:

$$\|(\mathscr{S} - \mathscr{S}_h)\mathbf{f}\|_{H^1(\Omega)^2} \leqslant ch^s(\|\mathbf{f}\|_{H^{s-1}(\Omega)^2} + \|\mathscr{S}\mathbf{f}\|_{H^{s+1}(\Omega)^2}) \tag{41}$$

Similarly, we introduce the space

$$\mathbb{Z}_h = \{t_h \in H^1(\Omega);\ \forall K \in \mathscr{T}_h,\ t_{h|K} \in \mathscr{P}_1(K)\} \tag{42}$$

where $\mathscr{P}_1(K)$ stands for the space of restrictions to K of affine functions on $\mathbb{R}^2$. Let $\mathscr{L}_\varepsilon$ denote the ε-dependent Laplace operator which associates with any data g in $H^1(\Omega)'$ the solution z in $H^1(\Omega)$ of the problem

$$\forall t \in H^1(\Omega), \quad c_\varepsilon(z,t) = \langle g, t\rangle \tag{43}$$

The approximation $\mathscr{L}_{\varepsilon h}$ of this operator is defined as follows: For any data g in $H^1(\Omega)'$, the function $z_h = \mathscr{L}_{\varepsilon h} g$ belongs to $\mathbb{Z}_h$ and satisfies

$$\forall t_h \in \mathbb{Z}_h, \quad c_\varepsilon(z_h, t_h) = \langle g, t_h\rangle \tag{44}$$

For completeness we recall two results which are completely standard for $\varepsilon = 1$.

(i) The following stability properties hold:

$$\|\mathscr{L}_{\varepsilon h} g\|_\varepsilon \leqslant c\varepsilon^{-1/2}\|g\|_{H^1(\Omega)'}, \quad \|\mathscr{L}_{\varepsilon h} g\|_\varepsilon \leqslant c\|g\|_{L^2(\Omega)} \tag{45}$$

(ii) If the solution $\mathscr{L}_\varepsilon g$ belongs to $H^{s+1}(\Omega)$, $0<s\leqslant 1$, the following error estimate holds:

$$\|(\mathscr{L}_\varepsilon - \mathscr{L}_{\varepsilon h})g\|_\varepsilon \leqslant ch^s(h + \varepsilon^{1/2})\|\mathscr{L}_\varepsilon g\|_{H^{s+1}(\Omega)} \tag{46}$$

The discrete problem associated with problem (16) now reads:

$$\begin{aligned}
&\text{Find } (\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h}) \text{ in } \mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Z}_h \text{ such that}\\
&\forall \mathbf{v}_h \in \mathbb{X}_h, \quad a(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{v}_h, p_{\varepsilon h}) + A(z_{\varepsilon h}; \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) = \int_\Omega \mathbf{f}\cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}\\
&\forall q_h \in \mathbb{M}_h, \quad b(\mathbf{u}_{\varepsilon h}, q_h) = 0\\
&\forall t_h \in \mathbb{Z}_h, \quad c_\varepsilon(z_{\varepsilon h}, t_h) + C(\mathbf{u}_{\varepsilon h}; z_{\varepsilon h}, t_h) = \int_\Omega (\operatorname{curl} \mathbf{f}) t_h \,\mathrm{d}\mathbf{x} + \nu \int_\Omega (\operatorname{curl}\ \mathbf{u}_{\varepsilon h}) t_h \,\mathrm{d}\mathbf{x}
\end{aligned} \tag{47}$$

It is readily checked that, for a fixed $z_{\varepsilon h}$ in $\mathbb{Z}_h$, the first two lines in (47) have a unique solution $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ and that the velocity $\mathbf{u}_{\varepsilon h}$ belongs to

$$\mathbb{V}_h = \{\mathbf{v}_h \in \mathbb{X}_h;\ \forall q_h \in \mathbb{M}_h,\ b(\mathbf{v}_h, q_h) = 0\} \tag{48}$$

However the study of the fully coupled problem (47) is more complex and requires a different formulation. Indeed, the analysis of this problem relies on the theorem due to Brezzi, Rappaz and Raviart [30] (see also Reference [31, Theorem 3.1] or Reference [20, Chapter IV, Theorem 3.1]). To apply it, we first introduce a new formulation of problem (17), which makes use of the operator $\mathscr{S}$ and $\mathscr{L}_\varepsilon$. We also define the mapping F from $H_0^1(\Omega)^2 \times L^2(\Omega)$ into $H^{-1}(\Omega)^2$ by

$$\forall U = (\mathbf{u}, z) \in H_0^1(\Omega)^2 \times L^2(\Omega),\ \forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \langle F(U), \mathbf{v}\rangle = A(z; \mathbf{u}, \mathbf{v}) - \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \tag{49}$$

and the mapping G from $H^s(\Omega)^2 \times H^1(\Omega)$, $1 < s < \frac{3}{2}$, into $L^2(\Omega)$ by

$\forall U = (\mathbf{u}, z) \in H^s(\Omega)^2 \times H^1(\Omega),\ \forall t \in L^2(\Omega),$

$$\langle G(U), t\rangle = C(\mathbf{u}; z, t) - \alpha \int_\Omega (\operatorname{curl} \mathbf{f}) t\, \mathrm{d}\mathbf{x} - v \int_\Omega (\operatorname{curl} \mathbf{u}) t\, \mathrm{d}\mathbf{x} \tag{50}$$

We define the space $\mathscr{Y}_\varepsilon$ as the product $H_0^1(\Omega)^2 \times H^1(\Omega)$, equipped with the norm

$$\|V\|_{\mathscr{Y}_\varepsilon} = (|\mathbf{v}|^2_{H^1(\Omega)^2} + \|t\|^2_\varepsilon)^{1/2} \quad \text{with } V = (\mathbf{v}, t) \tag{51}$$

With these notations, it is readily checked that the pair $U_\varepsilon = (\mathbf{u}_\varepsilon, z_\varepsilon)$ is a solution of problem (17) if and only if it belongs to $\mathscr{Y}_\varepsilon$ and satisfies

$$\mathscr{H}_\varepsilon(U_\varepsilon) = U_\varepsilon + \begin{pmatrix} \mathscr{S} & 0 \\ 0 & \mathscr{L}_\varepsilon \end{pmatrix} \begin{pmatrix} F(U_\varepsilon) \\ G(U_\varepsilon) \end{pmatrix} = 0 \tag{52}$$

Similarly, the pair $U_{\varepsilon h} = (\mathbf{u}_{\varepsilon h}, z_{\varepsilon h})$ is the part of a solution $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ of problem (47) if and only if it belongs to the discrete space $\mathscr{Y}_h = \mathbb{X}_h \times \mathbb{Z}_h$ and satisfies

$$\mathscr{H}_{\varepsilon h}(U_{\varepsilon h}) = U_{\varepsilon h} + \begin{pmatrix} \mathscr{S}_h & 0 \\ 0 & \mathscr{L}_{\varepsilon h} \end{pmatrix} \begin{pmatrix} F(U_{\varepsilon h}) \\ G(U_{\varepsilon h}) \end{pmatrix} = 0 \tag{53}$$

We now prove some properties of the previous mappings.

Lemma 4.1
Let U_ε be a solution of problem (52) in $\mathscr{Y}_\varepsilon$ such that the operator

$$\mathscr{G}_\varepsilon(U_\varepsilon) = \mathscr{I} + \begin{pmatrix} \mathscr{S} & 0 \\ 0 & \mathscr{L}_\varepsilon \end{pmatrix} \begin{pmatrix} DF(U_\varepsilon) \\ DG(U_\varepsilon) \end{pmatrix} \tag{54}$$

is an automorphism of $\mathscr{Y}_\varepsilon$, and let γ_ε denote the norm of its inverse. Then, there exists an $h_0 > 0$ only depending on ε and γ_ε such that, for all $h \leqslant h_0$, the operator

$$\mathscr{G}_{\varepsilon h}(U_\varepsilon) = \mathscr{I} + \begin{pmatrix} \mathscr{S}_h & 0 \\ 0 & \mathscr{L}_{\varepsilon h} \end{pmatrix} \begin{pmatrix} DF(U_\varepsilon) \\ DG(U_\varepsilon) \end{pmatrix} \tag{55}$$

is an automorphism of $\mathscr{Y}_\varepsilon$ with the norm of its inverse smaller than $2\gamma_\varepsilon$.

Proof
Thanks to the formula

$$\mathscr{G}_{\varepsilon h}(U_\varepsilon) = \mathscr{G}_\varepsilon(U_\varepsilon) \left(\mathscr{I} - \mathscr{G}_\varepsilon^{-1}(U_\varepsilon) \begin{pmatrix} \mathscr{S} - \mathscr{S}_h & 0 \\ 0 & \mathscr{L}_\varepsilon - \mathscr{L}_{\varepsilon h} \end{pmatrix} \begin{pmatrix} DF(U_\varepsilon) \\ DG(U_\varepsilon) \end{pmatrix} \right)$$

it suffices to check that the norm of the operator

$$\begin{pmatrix} \mathscr{S} - \mathscr{S}_h & 0 \\ 0 & \mathscr{L}_\varepsilon - \mathscr{L}_{\varepsilon h} \end{pmatrix} \begin{pmatrix} DF(U_\varepsilon) \\ DG(U_\varepsilon) \end{pmatrix}$$

in the space $\mathscr{L}(\mathscr{Y}_\varepsilon)$ of linear mappings from $\mathscr{Y}_\varepsilon$ into itself tends to 0 when h tends to zero. This follows from (41) and (46) combined with the compactness of $DF(U_\varepsilon)$ from $\mathscr{Y}_\varepsilon$ into $H^{-1}(\Omega)^2$ and of $DG(U_\varepsilon)$ from $\mathscr{Y}_\varepsilon$ into $H^1(\Omega)'$, both being a consequence of the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$.

Lemma 4.2
There exists a constant λ independent of ε such that the following Lipschitz property holds for any solution U_ε of problem (52) in $\mathscr{Y}_\varepsilon$ and any $\tilde{U}$ in $\mathscr{Y}_\varepsilon$ such that $\|U_\varepsilon - \tilde{U}\|_{\mathscr{Y}_\varepsilon} \leqslant \tau$:

$$\|\mathscr{G}_{\varepsilon h}(\tilde{U}) - \mathscr{G}_{\varepsilon h}(U_\varepsilon)\|_{\mathscr{L}(\mathscr{Y}_\varepsilon)} \leqslant \lambda \varepsilon^{-1/2} |\log \varepsilon|^{1/2} \tau. \tag{56}$$

Proof
We prove the Lipschitz property successively for the two lines of $\mathscr{K}_{\varepsilon h}$.

(1) By combining the stability property (40) and the fact that the mapping $\tilde{U} \mapsto DF(\tilde{U})$ is linear, we easily derive with obvious notation

$$\|\mathscr{S}_h(DF(\tilde{U}) - DF(U_\varepsilon))\|_{\mathscr{L}(\mathscr{Y}_\varepsilon, H_0^1(\Omega)^2)} \leqslant c(|\tilde{\mathbf{u}} - \mathbf{u}_\varepsilon|_{H^1(\Omega)^2} + \|\tilde{z} - z_\varepsilon\|_{L^2(\Omega)})$$

(2) Here we have, for $W = (w, r)$,

$$\forall t \in H^1(\Omega), \quad \langle DG(\tilde{U}).W, t\rangle = C(\mathbf{w}; \tilde{z}, t) + C(\tilde{\mathbf{u}}; r, t) - v \int_\Omega (\operatorname{curl} \mathbf{w}) t \, d\mathbf{x}$$

whence

$$\forall t \in H^1(\Omega), \quad \langle (DG(\tilde{U}) - DG(U_\varepsilon)).W, t\rangle = C(\mathbf{w}; \tilde{z} - z_\varepsilon, t) + C(\tilde{\mathbf{u}} - \mathbf{u}_\varepsilon; r, t)$$

Using once more the fact [27] that the norm of the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is smaller than $c p^{1/2}$ and defining p' such that $1/p + 1/p' = \frac{1}{2}$, we have

$$|C(\mathbf{w}; \tilde{z} - z_\varepsilon, t)| \leqslant c p^{1/2} |\mathbf{w}|_{H_0^1(\Omega)^2} \varepsilon^{-1/2} \|\tilde{z} - z_\varepsilon\|_\varepsilon \|t\|_{L^{p'}(\Omega)}$$

Assuming without restriction that p is >4 and noting that the norm of the imbedding of $H^s(\Omega)$ with $s=\frac{2}{p}$ into $L^{p'}(\Omega)$ is bounded independently of p (this follows by an interpolation argument), we obtain

$$|C(\mathbf{w};\tilde{z}-z_\varepsilon,t)|\leqslant c\,p^{1/2}|\mathbf{w}|_{H_0^1(\Omega)^2}\varepsilon^{-1/2}\|\tilde{z}-z_\varepsilon\|_\varepsilon\|t\|_{H^s(\Omega)}$$

and a similar bound holds for the term $C(\tilde{\mathbf{u}}-\mathbf{u}_\varepsilon;r,t)$. On the other hand, it can be noted that, since s is $<\frac{1}{2}$, the spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ coincide. So, an interpolation argument relying on Reference [32, Chapitre I, Théorème 6.2] between the two inequalities in (45) leads to

$$\|\mathscr{L}_{\varepsilon h}g\|_\varepsilon\leqslant c\varepsilon^{-s/2}\|g\|_{H^{-s}(\Omega)}$$

Combining all this yields

$$\|\mathscr{L}_{\varepsilon h}(DG(\tilde{U})-DG(U_\varepsilon)).W\|_\varepsilon\leqslant c\,p^{1/2}\varepsilon^{-1/2-1/p}\ (|\mathbf{w}|_{H_0^1(\Omega)^2}\|\tilde{z}-z_\varepsilon\|_\varepsilon+|\tilde{\mathbf{u}}-\mathbf{u}_\varepsilon|_{H_0^1(\Omega)^2}\|r\|_\varepsilon)$$

We conclude the proof by taking $p=c|\log\varepsilon|$.

Lemma 4.3
For any data $\mathbf{f}$ in $H(\mathrm{curl},\Omega)$, let U_ε be a solution of problem (52) in $H^{s+1}(\Omega)^2\times H^{s+1}(\Omega)$, $0<s\leqslant 1$. The following estimate holds for a constant $c(U_\varepsilon,\mathbf{f})$ only depending on the norms of U_ε and $\mathbf{f}$ in these spaces

$$\|\mathscr{H}_{\varepsilon h}(U_\varepsilon)\|_{\mathscr{Y}_\varepsilon}\leqslant c(U_\varepsilon,\mathbf{f})h^s \tag{57}$$

Proof
This follows from the formula

$$\mathscr{H}_{\varepsilon h}(U_\varepsilon)=-\begin{pmatrix}\mathscr{S}-\mathscr{S}_h & 0\\ 0 & \mathscr{L}_\varepsilon-\mathscr{L}_{\varepsilon h}\end{pmatrix}\begin{pmatrix}F(U_\varepsilon)\\ G(U_\varepsilon)\end{pmatrix}$$

together with (41) and (46).

Remark 4.4
A more precise application of (41) and (46), combined with Lemma 3.1 leads to the improved (but less simple) estimate

$$\|\mathscr{H}_{\varepsilon h}(U_\varepsilon)\|_{\mathscr{Y}_\varepsilon}\leqslant ch^s\|\mathbf{u}_\varepsilon\|_{H^{s+1}(\Omega)^2}(1+\|\mathbf{f}\|_{H(\mathrm{curl},\Omega)})+c'h^s(h+\varepsilon^{1/2})\|z_\varepsilon\|_{H^{s+1}(\Omega)} \tag{58}$$

We are now in a position to state and prove the main result of this section.

Theorem 4.5
For any data $\mathbf{f}$ in $H(\mathrm{curl},\Omega)$, let $(\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ be a solution of problem (16) such that the pair $U_\varepsilon=(\mathbf{u}_\varepsilon,z_\varepsilon)$ belongs to $H^{s+1}(\Omega)^2\times H^{s+1}(\Omega)$, $0<s\leqslant 1$ and the operator $\mathscr{K}_\varepsilon(U_\varepsilon)$ defined in (54) is an automorphism of $\mathscr{Y}_\varepsilon$. Then, there exist two positive constants h_0^* and μ only depending on ε and the constant γ_ε defined in Lemma 4.1 such that, for all $h\leqslant h_0^*$, problem (47) has a unique solution $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ satisfying

$$\|U_\varepsilon-U_{\varepsilon h}\|_{\mathscr{Y}_\varepsilon}\leqslant\mu\quad\text{with } U_{\varepsilon h}=(\mathbf{u}_{\varepsilon h},z_{\varepsilon h}) \tag{59}$$

Moreover, the following *a priori* error estimate holds for a constant $c(U_\varepsilon, \mathbf{f})$ only depending on the norms of U_ε and $\mathbf{f}$ in the previous spaces

$$|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(\Omega)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon \leqslant c(U_\varepsilon, \mathbf{f})h^s \tag{60}$$

Proof

Since the assumptions of the Brezzi–Rappaz–Raviart theorem [30] can be derived from Lemmas 4.1–4.3, applying this theorem yields the existence and uniqueness of a solution $U_{\varepsilon h}$ of problem (53) satisfying (59), together with the error estimate for $\|U_\varepsilon - U_{\varepsilon h}\|_{\mathscr{Y}_\varepsilon}$. The existence of a discrete pressure $p_{\varepsilon h}$ and the estimate for $\|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)}$ are then an obvious consequence of the inf–sup condition (39).

We note that the constant μ in (59) must only satisfy, for a constant c depending on the geometry of Ω and the regularity parameter σ,

$$4\gamma_\varepsilon \varepsilon^{-1/2} |\log \varepsilon|^{1/2} \mu < c \tag{61}$$

This upper bound limits ε, but the limitation is completely independent of h. The estimate (60) is optimal. Moreover, by replacing its right-hand side by that of (58) and combining this with (22), we obtain the following result.

Corollary 4.6

If the assumptions of Theorem 4.5 are satisfied, the following *a priori* error estimate holds for a constant $c(\mathbf{f})$ only depending on the norm of $\mathbf{f}$ in $H(\text{curl}, \Omega)$ and for the values of s indicated in Proposition 3.4:

$$|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(\Omega)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon \leqslant c(\mathbf{f})h^s(1 + h\varepsilon^{-s-1/2} + \varepsilon^{-s}) \tag{62}$$

As a consequence and in the case of a convex domain for instance, taking h of the same order as $\varepsilon^{3/2}$ leads to a convergence of order $\varepsilon^{1/2}$ or $h^{1/3}$.

5. *A POSTERIORI* ANALYSIS

Since a non-discrete intermediate problem, namely problem (16), is introduced between the continuous problem (4) and the discrete problem (47), the *a posteriori* analysis relies on the ideas presented in Reference [33] in a similar case. More precisely, we consider two types of error indicators: One is linked to the regularization step and the other ones, which are linked to the finite element discretization, are the sum of two parts which correspond to the equation on $(\mathbf{u}, p)$ and z, respectively. Indeed the aim of this is to optimize the choice of the parameter ε when working with adaptive meshes. We separately introduce the two types of error indicators and prove upper and lower bounds for each of them. We conclude with a global *a posteriori* error estimate.

From now on, we fix an approximation $\mathbf{f}_h$ of the data $\mathbf{f}$ in the space associated with the Raviart–Thomas finite element [34]

$$\mathbb{T}_h = \{\mathbf{k}_h \in H(\text{curl}, \Omega);\ \forall K \in \mathscr{T}_h, \mathbf{k}_{h|K} \in \mathscr{P}_{RT}(K)\} \tag{63}$$

where $\mathscr{P}_{RT}(K)$ stands for the space of restrictions to K of polynomials of the form $\mathbf{a} + b \times \mathbf{x}$ (here $\mathbf{x}$ is the vector with components x and y).

5.1. A posteriori estimate of the regularization error

The error indicator η_ε is defined by

$$\eta_\varepsilon = \|\alpha\,\mathrm{curl}\,\mathbf{f}_h + v\,\mathrm{curl}\,\mathbf{u}_{\varepsilon h} - v z_{\varepsilon h} - \alpha\mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(\Omega)} \tag{64}$$

It only deals with the residual of the equation on z and is global on the whole domain Ω. Note that it is very easy to compute once the discrete solution is known since the norm in $L^2(\Omega)$ which appears in its definition is the Hilbertian sum of norms in $L^2(K)$ of quantities which are quadratic on K.

The *a posteriori* error estimate is simply derived by combining (33) and (34) with a triangle inequality. From now on, we denote by $h_{\min}$ the smallest diameter of the triangles K in $\mathscr{T}_h$.

Proposition 5.1

If the assumptions of Proposition 3.9 and Theorem 4.5 hold, there exists a constant $c_1(U)$ only depending on U and s and a constant $c_2(\mathbf{f})$ only depending on the data $\mathbf{f}$ such that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{(H^1(\Omega)\cap L^\infty(\Omega))^2} + \|p-p_\varepsilon\|_{L^2(\Omega)} + \|z-z_\varepsilon\|_{L^2(\Omega)}\\ \leqslant c_1(U)(\eta_\varepsilon + c_2(\mathbf{f})\varepsilon^{-1/2}(|\log h_{\min}|^{1/2}\|\mathbf{u}_\varepsilon-\mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon)\\ +\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(\Omega)})\end{aligned} \tag{65}$$

where $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ denotes the unique solution of problem (47) satisfying (59).

Proof

Thanks to (33) and (34), the term in the left-hand side of (65) is smaller than $(C(U)+C'(U))\varepsilon\|\Delta z_\varepsilon\|_{L^2(\Omega)}$. To bound this last quantity, we first observe from (15) that

$$\varepsilon\|\Delta z_\varepsilon\|_{L^2(\Omega)} = \|\alpha\,\mathrm{curl}\,\mathbf{f} + v\,\mathrm{curl}\,\mathbf{u}_\varepsilon - v z_\varepsilon - \alpha\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon\|_{L^2(\Omega)}$$

whence

$$\begin{aligned}\varepsilon\|\Delta z_\varepsilon\|_{L^2(\Omega)} \leqslant \eta_\varepsilon + |\alpha|\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(\Omega)} + v\|\mathrm{curl}(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})\|_{L^2(\Omega)}\\ +v\|z_\varepsilon - z_{\varepsilon h}\|_{L^2(\Omega)} + |\alpha|\|\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon - \mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(\Omega)}\end{aligned}$$

To estimate the nonlinear term, we use the further triangle inequality, for any $p>2$ and with $1/p + 1/p' = \frac{1}{2}$,

$$\|\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon - \mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(\Omega)} \leqslant \|\mathbf{u}_\varepsilon\|_{L^\infty(\Omega)^2}\|\nabla(z_\varepsilon - z_{\varepsilon h})\|_{L^2(\Omega)^2} + \|\nabla z_{\varepsilon h}\|_{L^{p'}(\Omega)^2}\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{L^p(\Omega)^2}$$

We use a further inverse inequality for $\|\nabla z_{\varepsilon h}\|_{L^{p'}(\Omega)^2}$, note that $\|\nabla z_{\varepsilon h}\|_{L^2(\Omega)^2}$ is bounded from (57). Thus, since the norm of the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ behaves like $p^{1/2}$, taking $p = |\log h_{\min}|$ gives the desired result.

The proof of the converse estimate relies on exactly the same triangle inequalities as previously, so we present it in an abridged way.

Proposition 5.2
If the assumptions of Proposition 3.9 and Theorem 4.5 hold, there exists a constant $c_3(U)$ only depending on U such that

$$\eta_\varepsilon \leqslant c_3(U)(\varepsilon^{-1/2}(\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{(H^1(\Omega)\cap L^\infty(\Omega))^2} + \|z-z_\varepsilon\|_\varepsilon + |\log h_{\min}|^{1/2}\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon) + \|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(\Omega)}) \tag{66}$$

where $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ denotes the unique solution of problem (47) satisfying (59).

Proof
The arguments used in the proof of Proposition 5.1 yield that

$$\begin{aligned}\eta_\varepsilon \leqslant{}& \|\alpha\,\mathrm{curl}\,\mathbf{f} + v\,\mathrm{curl}\,\mathbf{u}_\varepsilon - vz_\varepsilon - \alpha\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon\|_{L^2(\Omega)}\\ &+ c_2(\mathbf{f})\varepsilon^{-1/2}(|\log h_{\min}|^{1/2}\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon)\\ &+\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(\Omega)}\end{aligned}$$

Next, replacing $\alpha\,\mathrm{curl}\,\mathbf{f}$ by $-v\,\mathrm{curl}\,\mathbf{u} + vz + \alpha\mathbf{u}\cdot\nabla z$ gives

$$\begin{aligned}&\|\alpha\,\mathrm{curl}\,\mathbf{f} + v\,\mathrm{curl}\,\mathbf{u}_\varepsilon - vz_\varepsilon - \alpha\mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon\|_{L^2(\Omega)}\\ &\quad\leqslant c(\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{H^1(\Omega)^2} + \|z-z_\varepsilon\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla z - \mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon\|_{L^2(\Omega)})\end{aligned}$$

Using the next inequality to bound the nonlinear term

$$\|\mathbf{u}\cdot\nabla z - \mathbf{u}_\varepsilon\cdot\nabla z_\varepsilon\|_{L^2(\Omega)} \leqslant \|\mathbf{u}\|_{L^\infty(\Omega)^2}\|\nabla(z-z_\varepsilon)\|_{L^2(\Omega)^2} + \|\nabla z_\varepsilon\|_{L^2(\Omega)^2}\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{L^\infty(\Omega)^2}$$

leads to the desired estimate.

5.2. *A posteriori estimate of the finite element error*

We first introduce some notation: For any K in $\mathcal{T}_h$, we denote by h_K the diameter of K, by $\mathcal{E}_K$ the set of the three edges of K and by $\mathcal{E}_K^0$ the subset of $\mathcal{E}_K$ consisting of all edges that are not contained in $\partial\Omega$. For each element e of $\mathcal{E}_K$, h_e stands for the length of e and, for any function v, the quantity $[v]_e$ denotes the trace of v on e if e is contained in $\partial\Omega$, the jump of v through e otherwise (making the sign precise is irrelevant in what follows).

With each K in $\mathcal{T}_h$, we associate two error indicators, related to the residuals of the equations on $(\mathbf{u}, p)$ and z, respectively:

- Error indicators linked to $(\mathbf{u}, p)$.
  These indicators are of standard residual type, see Reference [16, Section 2.1]. For each K in $\mathcal{T}_h$, the indicator $\eta_{K\sharp}$ is defined by

$$\begin{aligned}\eta_{K\sharp} ={}& h_K\|\mathbf{f}_h + v\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h}\|_{L^2(K)^2}\\ &+\sum_{e\in\mathcal{E}_K^0} h_e^{1/2}\|[v\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\|_{L^2(e)^2} + \|\mathrm{div}\,\mathbf{u}_{\varepsilon h}\|_{L^2(K)}\end{aligned} \tag{67}$$

Note that the two lines in this definition correspond to the residuals of the first two lines in problem (2) since **grad** $p_{\varepsilon h}$ is zero on each triangle.

- Error indicators linked to z.

  These indicators are rather similar to those introduced in Reference [18] for an uncoupled convection–diffusion equation. For each K in $\mathscr{T}_h$, the indicator $\eta_{K\flat}$ is defined by

$$\eta_{K\flat} = \varepsilon^{-1/2} h_K \|\alpha\,\mathrm{curl}\,\mathbf{f}_h + v\,\mathrm{curl}\,\mathbf{u}_{\varepsilon h} - v z_{\varepsilon h} - \alpha \mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(K)} + \varepsilon^{1/2}\sum_{e\in\mathscr{E}_K} h_e^{1/2}\|[\partial_{n_K} z_{\varepsilon h}]_e\|_{L^2(e)} \tag{68}$$

To prove the *a posteriori* error estimate, we need a formulation of (16) which is slightly different from that introduced in Section 4; see (52). Indeed, let $\mathscr{S}^*$ denote the generalized Stokes operator which, with any data $(\mathbf{f},\ell)$ in $H^{-1}(\Omega)^2\times L_0^2(\Omega)$, associates the solution $(\mathbf{u}, p)$ in $H_0^1(\Omega)^2\times L_0^2(\Omega)$ of the equations

$$\begin{aligned}&\forall \mathbf{v}\in H_0^1(\Omega)^2, \quad a(\mathbf{u},\mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f},\mathbf{v}\rangle\\ &\forall q\in L_0^2(\Omega), \quad b(\mathbf{u},q) = \langle \ell, q\rangle\end{aligned} \tag{69}$$

We also define the space $\mathscr{Y}_\varepsilon^*$ as the product $H_0^1(\Omega)^2\times L_0^2(\Omega)\times H^1(\Omega)$ and we observe that a triple $U_\varepsilon^* = (\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ is a solution to problem (16) if and only if it belongs to $\mathscr{Y}_\varepsilon^*$ and satisfies (with obvious notation for U_ε)

$$\mathscr{H}_\varepsilon^*(U_\varepsilon^*) = U_\varepsilon^* + \begin{pmatrix}\mathscr{S}^* & 0\\ 0 & \mathscr{L}_\varepsilon\end{pmatrix}\begin{pmatrix}F(U_\varepsilon)\\ 0\\ G(U_\varepsilon)\end{pmatrix} = 0 \tag{70}$$

where F and G are defined in (49) and (50), respectively. Similarly, any $U_{\varepsilon h}^* = (\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ which is a solution to problem (47) satisfies

$$\mathscr{H}_\varepsilon^*(U_{\varepsilon h}^*) = \begin{pmatrix}\mathscr{S}^* & 0\\ 0 & \mathscr{L}_\varepsilon\end{pmatrix}\begin{pmatrix}R_1(U_{\varepsilon h}^*)\\ R_2(U_{\varepsilon h}^*)\\ R_3(U_{\varepsilon h}^*)\end{pmatrix}$$

where the residuals R_i are defined by

$$\forall \mathbf{v}\in H_0^1(\Omega)^2, \quad \langle R_1(U_{\varepsilon h}^*),\mathbf{v}\rangle = a(\mathbf{u}_{\varepsilon h},\mathbf{v}) + b(\mathbf{v}, p_{\varepsilon h}) + A(z_{\varepsilon h};\mathbf{u}_{\varepsilon h},\mathbf{v}) - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

$$\forall q\in L^2(\Omega), \quad \langle R_2(U_{\varepsilon h}^*), q\rangle = b(\mathbf{u}_{\varepsilon h}, q)$$

and

$$\forall t\in H^1(\Omega), \quad \langle R_3(U_{\varepsilon h}^*), t\rangle = c_\varepsilon(z_{\varepsilon h}, t) + C(\mathbf{u}_{\varepsilon h}; z_{\varepsilon h}, t) - \alpha\int_\Omega (\mathrm{curl}\,\mathbf{f})t\,\mathrm{d}\mathbf{x} - v\int_\Omega (\mathrm{curl}\,\mathbf{u}_{\varepsilon h})t\,\mathrm{d}\mathbf{x}$$

Proposition 5.3
Let $U_\varepsilon^* = (\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ be a solution of problem (16) in $\mathscr{Y}_\varepsilon^*$ such that the operator $D\mathscr{H}_\varepsilon^*(U_\varepsilon^*)$ is an automorphism of $\mathscr{Y}_\varepsilon^*$. There exist a neighbourhood $\mathscr{U}_\varepsilon^*$ of U_ε^* in $\mathscr{Y}_\varepsilon^*$ and a constant $c_3(U_\varepsilon^*)$ depending only on U_ε^* such that any solution $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}, z_{\varepsilon h})$ of problem (47) which belongs to $\mathscr{U}_\varepsilon^*$ satisfies

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon$$
$$\leqslant c_3(U_\varepsilon^*)\left(\sum_{K\in\mathscr{T}_h}(\eta_{K\sharp}^2 + \eta_{K\flat}^2 + h_K^2\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}^2 + h_K^2\varepsilon^{-1}\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(K)}^2)\right)^{1/2} \quad (71)$$

Proof
The Lipschitz property of the mapping: $\tilde{U}^* \mapsto D\mathscr{H}_\varepsilon^*(\tilde{U}^*)$ in $\mathscr{L}(\mathscr{Y}_\varepsilon^*)$ can easily be verified in a neighbourhood of U_ε^* thanks to the same arguments as in Lemma 3.6. Then, applying Reference [16, Proposition 2.1] leads to the following estimate:

$$\|U_\varepsilon^* - U_{\varepsilon h}^*\|_{\mathscr{Y}_\varepsilon^*} \leqslant c(U_\varepsilon^*)\left\|\begin{pmatrix}\mathscr{S}^* & 0\\ 0 & \mathscr{L}_\varepsilon\end{pmatrix}\begin{pmatrix}R_1(U_{\varepsilon h}^*)\\ R_2(U_{\varepsilon h}^*)\\ R_3(U_{\varepsilon h}^*)\end{pmatrix}\right\|_{\mathscr{Y}_\varepsilon^*}$$

Combining this with the stability properties of the operator $\mathscr{S}^*$ and $\mathscr{L}_\varepsilon$ gives

$$\|U_\varepsilon^* - U_{\varepsilon h}^*\|_{\mathscr{Y}_\varepsilon^*} \leqslant c(U_\varepsilon^*)(\|R_1(U_{\varepsilon h}^*)\|_{H^{-1}(\Omega)^2} + \|R_2(U_{\varepsilon h}^*)\|_{L^2(\Omega)} + \varepsilon^{-1/2}\|R_3(U_{\varepsilon h}^*)\|_{H^1(\Omega)'})$$

We now bound successively the three residual terms $R_i(U_{\varepsilon h}^*)$.

(1) Noting that, for any $\mathbf{v}$ in $H_0^1(\Omega)^2$ and $\mathbf{v}_h$ in $\mathbb{X}_h$,

$$\langle R_1(U_{\varepsilon h}^*), \mathbf{v}\rangle = \langle R_1(U_{\varepsilon h}^*), \mathbf{v}-\mathbf{v}_h\rangle$$

integrating by parts on each K in the right-hand side of this equation and taking $\mathbf{v}_h$ equal to the image of $\mathbf{v}$ by a Clément type regularization operator (see Reference [20, Chapter I, Theorem A.4]), we derive

$$\|R_1(U_{\varepsilon h}^*)\|_{H^{-1}(\Omega)^2} \leqslant c\left(\sum_{K\in\mathscr{T}_h}\left(h_K\|\mathbf{f} + \nu\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h}\|_{L^2(K)^2} + \sum_{e\in\mathscr{E}_K^0} h_e^{1/2}\|[\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\|_{L^2(e)^2}\right)^2\right)^{1/2}$$

(2) A Cauchy–Schwarz inequality leads to

$$\|R_2(U_{\varepsilon h}^*)\|_{L^2(\Omega)} \leqslant \|\mathrm{div}\,\mathbf{u}_{\varepsilon h}\|_{L^2(\Omega)}$$

(3) We note that, for any t in $H^1(\Omega)$ and t_h in $\mathbb{Z}_h$,

$$\langle R_3(U^*_{\varepsilon h}), t\rangle = \langle R_3(U^*_{\varepsilon h}), t - t_h\rangle$$

whence, by integration by parts on each K and noting that $\Delta z_{\varepsilon h}$ is zero on each K,

$$\langle R_3(U^*_{\varepsilon h}), t\rangle \leqslant \sum_{K\in\mathscr{T}_h}\Bigg(\|\alpha\,\mathrm{curl}\,\mathbf{f} + \nu\,\mathrm{curl}\,\mathbf{u}_{\varepsilon h} - \nu z_{\varepsilon h} - \alpha\mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(K)}\|t - t_h\|_{L^2(K)} + \varepsilon\sum_{e\in\mathscr{E}_K}\|[\partial_{n_K} z_{\varepsilon h}]_e\|_{L^2(e)}\|t - t_h\|_{L^2(e)}\Bigg)$$

Also, taking t_h equal to the image of t by a regularization operator yields

$$\|R_3(U^*_{\varepsilon h})\|_{H^1(\Omega)'} \leqslant c\Bigg(\sum_{K\in\mathscr{T}_h}(h_K^2\|\alpha\,\mathrm{curl}\,\mathbf{f} + \nu\,\mathrm{curl}\,\mathbf{u}_{\varepsilon h} - \nu z_{\varepsilon h} - \alpha\mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|^2_{L^2(\Omega)} + \varepsilon\sum_{e\in\mathscr{E}_K} h_e\|[\partial_{n_K} z_{\varepsilon h}]_e\|^2_{L^2(e)})\Bigg)^{1/2}$$

We conclude by combining all these estimates and using a triangle inequality for the terms involving the data $\mathbf{f}$.

Remark 5.4
It is readily checked from the inf–sup condition (9) that, for any solution $U^*_\varepsilon = (\mathbf{u}_\varepsilon, p_\varepsilon, z_\varepsilon)$ of problem (16) in $\mathscr{Y}^*_\varepsilon$, the pair $U_\varepsilon = (\mathbf{u}_\varepsilon, z_\varepsilon)$ is a solution of problem (52) and moreover that the operator $D\mathscr{H}^*_\varepsilon(U^*_\varepsilon)$ is an automorphism of $\mathscr{Y}^*_\varepsilon$ if and only if $\mathscr{G}_\varepsilon(U_\varepsilon) = D\mathscr{H}_\varepsilon(U_\varepsilon)$ is an automorphism of $\mathscr{Y}_\varepsilon$. So the assumptions of Theorem 4.5 and Proposition 5.3 are fully equivalent.

We now prove upper bounds on the indicators $\eta_{K\sharp}$ and $\eta_{K\flat}$. The first estimate relies on the residual equations

$$\forall \mathbf{v}\in H^1_0(\Omega)^2,$$

$$a(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}, \mathbf{v}) + b(\mathbf{v}, p_\varepsilon - p_{\varepsilon h}) + A(z_\varepsilon; \mathbf{u}_\varepsilon, \mathbf{v}) - A(z_{\varepsilon h}; \mathbf{u}_{\varepsilon h}, \mathbf{v})$$

$$= \sum_{K\in\mathscr{T}_h}\left(\int_K(\mathbf{f} + \nu\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \frac{1}{2}\sum_{e\in\mathscr{E}^0_K}\int_e[\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\cdot\mathbf{v}\,\mathrm{d}\tau\right) \quad (72)$$

and

$$\forall q\in L^2(\Omega),\quad b(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}, q) = \int_\Omega(\mathrm{div}\,\mathbf{u}_{\varepsilon h})q\,\mathrm{d}\mathbf{x} \quad (73)$$

Since the proof of the next proposition is completely standard, see Reference [16, Chapter 3], we only recall the main arguments. Let ω_K denote the union of elements of $\mathscr{T}_h$ that share at least an edge with K.

Proposition 5.5
There exists a constant $c_4(\mathbf{f})$ only depending on the data $\mathbf{f}$ such that the following estimate holds for each indicator $\eta_{K\sharp}$, $K \in \mathscr{T}_h$:

$$\eta_{K\sharp} \leqslant c_4(\mathbf{f})(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\omega_K)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\omega_K)} + \|z_\varepsilon - z_{\varepsilon h}\|_{L^2(\omega_K)} + h_K\|\mathbf{f} - \mathbf{f}_h\|_{L^2(\omega_K)^2}) \quad (74)$$

Proof
The estimate is derived from two successive choices of the function $\mathbf{v}$ in (72) and one choice of the function q in (73).

(1) We first take $\mathbf{v}$ in (72) equal to

$$\mathbf{v}_K = \begin{cases} (\mathbf{f}_h + v\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h} \times \mathbf{u}_{\varepsilon h})\psi_K & \text{on } K \\ \mathbf{0} & \text{elsewhere} \end{cases}$$

where ψ_K denotes the bubble function on K (equal for instance to the product of the three barycentric coordinates on K). This gives (the norm of the embedding of $H_0^1(K)$ into $L^4(K)$ is evaluated by switching to a reference triangle)

$$\begin{aligned} \|(\mathbf{f}_h + v\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h} \times \mathbf{u}_{\varepsilon h})\psi_K^{1/2}\|^2_{L^2(K)^2} \leqslant{}& c(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K)} \\ &+ h_K^{1/2}\|\mathbf{u}_\varepsilon \times z_\varepsilon - \mathbf{u}_{\varepsilon h} \times z_{\varepsilon h}\|_{L^{4/3}(K)^2})|\mathbf{v}_K|_{H^1(K)^2} \\ &+ \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^2}\|\mathbf{v}_K\|_{L^2(K)^2} \end{aligned}$$

Since $\mathbf{w} = \mathbf{f}_h + v\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h} \times \mathbf{u}_{\varepsilon h}$ is a polynomial of degree $\leqslant 3$ on K, we have the standard direct and inverse inequalities

$$\|\mathbf{v}_K\|_{L^2(K)^2} \leqslant \|\mathbf{w}\|_{L^2(K)^2} \leqslant c\|\mathbf{w}\psi_K^{1/2}\|_{L^2(K)^2} \quad \text{and} \quad |\mathbf{v}_K|_{H^1(K)^2} \leqslant ch_K^{-1}\|\mathbf{v}_K\|_{L^2(K)^2}$$

These give

$$\begin{aligned} \|\mathbf{f}_h + v\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h} \times \mathbf{u}_{\varepsilon h}\|_{L^2(K)^2} \leqslant{}& ch_K^{-1}(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K)} \\ &+ ch_K^{1/2}\|\mathbf{u}_\varepsilon \times z_\varepsilon - \mathbf{u}_{\varepsilon h} \times z_{\varepsilon h}\|_{L^{4/3}(K)^2}) + \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^2} \end{aligned}$$

To evaluate the nonlinear term, we apply the triangle inequality

$$\|\mathbf{u}_\varepsilon \times z_\varepsilon - \mathbf{u}_{\varepsilon h} \times z_{\varepsilon h}\|_{L^{4/3}(K)^2} \leqslant \|\mathbf{u}_\varepsilon\|_{L^4(K)^2}\|z_\varepsilon - z_{\varepsilon h}\|_{L^2(K)} + \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{L^4(K)^2}\|z_{\varepsilon h}\|_{L^2(K)}$$

Now, using the fact that the norm of the embedding of $H^1(K)$ into $L^4(K)$ behaves like $ch_K^{-1/2}$ together with the stability properties (19) and (59), we derive

$$\|\mathbf{u}_\varepsilon \times z_\varepsilon - \mathbf{u}_{\varepsilon h} \times z_{\varepsilon h}\|_{L^{4/3}(K)^2} \leqslant c(\mathbf{f})h_K^{-1/2}(\|z_\varepsilon - z_{\varepsilon h}\|_{L^2(K)} + \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K)^2})$$

Combining all these and multiplying by h_K gives

$$h_K\|\mathbf{f}_h + \nu\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h}\|_{L^2(K)^2}$$
$$\leqslant c(\mathbf{f})(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K)} + \|z_\varepsilon - z_{\varepsilon h}\|_{L^2(K)} + h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}) \quad (75)$$

(2) Let e be an edge in $\mathscr{E}_K^0$, and let K' denote the other triangle of $\mathscr{T}_h$ that contains e. The idea here is to choose $\mathbf{v}$ in (72) equal to

$$\mathbf{v}_e = \begin{cases} \mathscr{R}_{e,K}([\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\psi_e) & \text{on } K \\ \mathscr{R}_{e,K'}([\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\psi_e) & \text{on } K' \\ \mathbf{0} & \text{elsewhere} \end{cases}$$

where ψ_e is now the bubble function on e and $\mathscr{R}_{e,K}$ and $\mathscr{R}_{e,K'}$ are lifting operators, constructed by affine transformation from a fixed lifting operator that, for a reference triangle $\hat{K}$ with edge $\hat{e}$, maps polynomials in $H_0^1(\hat{e})$ into polynomials in $H^1(\hat{K})$ which vanish on $\partial\hat{K}\backslash\hat{e}$. Thus, we have

$$\|[\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\psi_e^{1/2}\|_{L^2(e)^2}^2$$
$$\leqslant c(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K\cup K')^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K\cup K')}$$
$$+ c' h_K^{1/2}\|\mathbf{u}_\varepsilon\times z_\varepsilon - \mathbf{u}_{\varepsilon h}\times z_{\varepsilon h}\|_{L^{4/3}(K\cup K')^2})|\mathbf{v}_e|_{H^1(K\cup K')^2}$$
$$+ (\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K\cup K')^2} + \|\mathbf{f}_h + \nu\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h}\|_{L^2(K\cup K')^2})\|\mathbf{v}_e\|_{L^2(K\cup K')^2}$$

Using the same direct and inverse inequalities as previously, together with the stability properties valid for any polynomial w in $H_0^1(K)$

$$\|\mathscr{R}_{e,K}w\|_{L^2(K)} + h_K|\mathscr{R}_{e,K}w|_{H^1(K)} \leqslant c h_e^{1/2}\|w\|_{L^2(e)}$$

and their analogues on K', we obtain

$$\|[\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\|_{L^2(e)^2}^2 \leqslant c(h_e^{-1/2}(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K\cup K')^2}$$
$$+\|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K\cup K')} + h_K^{1/2}\|\mathbf{u}_\varepsilon\times z_\varepsilon - \mathbf{u}_{\varepsilon h}\times z_{\varepsilon h}\|_{L^{4/3}(K\cup K')^2})$$
$$+ h_e^{1/2}(\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K\cup K')^2} + \|\mathbf{f}_h + \nu\Delta\mathbf{u}_{\varepsilon h} - z_{\varepsilon h}\times\mathbf{u}_{\varepsilon h}\|_{L^2(K\cup K')^2}))\|\mathbf{v}_e\|_{L^2(e)^2}$$

The same arguments as in the first part of the proof for evaluating the nonlinear term, combined with (75) then lead to

$$h_e^{1/2}\|[\nu\partial_{n_K}\mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\mathbf{n}_K]_e\|_{L^2(e)^2} \leqslant c(\mathbf{f})(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K\cup K')^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(K\cup K')}$$
$$+\|z_\varepsilon - z_{\varepsilon h}\|_{L^2(K\cup K')} + h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K\cup K')^2}) \quad (76)$$

(3) Finally, taking q in (73) equal to

$$q_K = (\operatorname{div} u_{\varepsilon h})\chi_K$$

where χ_K denotes the characteristic function of K leads in an obvious way to

$$\|\operatorname{div} \mathbf{u}_{\varepsilon h}\|_{L^2(K)} \leqslant c|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(K)^2} \tag{77}$$

Combining estimates (75) to (77) gives (74).

Estimating $\eta_{K\flat}$ relies on the same arguments, now combined with the residual equation

$$\forall t \in H^1(\Omega), \quad c_\varepsilon(z_\varepsilon - z_{\varepsilon h}, t) + C(\mathbf{u}_\varepsilon; z_\varepsilon, t) - C(\mathbf{u}_{\varepsilon h}; z_{\varepsilon h}, t)$$

$$= \sum_{K \in \mathscr{T}_h} \left(\int_K (\alpha \operatorname{curl} \mathbf{f} + v \operatorname{curl} \mathbf{u}_\varepsilon - v z_{\varepsilon h} - \alpha \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h}) t \, d\mathbf{x} + \varepsilon \sum_{e \in \mathscr{E}_K} \delta_e \int_e [\partial_{n_K} z_{\varepsilon h}]_e t \, d\tau \right) \tag{78}$$

where δ_e is equal to 1 or $\frac{1}{2}$, depending on whether or not e is contained in $\partial\Omega$. Let $\|\cdot\|_{\varepsilon,\omega}$ stand for the restriction of the norm $\|\cdot\|_\varepsilon$ to any ω contained in Ω.

Proposition 5.6
There exists a constant $c_5(\mathbf{f})$ only depending on the data $\mathbf{f}$ such that the following estimate holds for each indicator $\eta_{K\flat}$, $K \in \mathscr{T}_h$:

$$\eta_{K\flat} \leqslant c_5(\mathbf{f})\varepsilon^{-1}(\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\omega_K)^2} + (\varepsilon + h_K)\|z_\varepsilon - z_{\varepsilon h}\|_{\varepsilon,\omega_K} + \varepsilon^{1/2} h_K \|\operatorname{curl}(\mathbf{f} - \mathbf{f}_h)\|_{L^2(\omega_K)^2}) \tag{79}$$

Proof
With the same notation as in the proof of Proposition 5.5, we now take t in (78) equal to

$$t_K = \begin{cases} (\alpha \operatorname{curl} \mathbf{f}_h + v \operatorname{curl} \mathbf{u}_{\varepsilon h} - v z_{\varepsilon h} - \alpha \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h})\psi_K & \text{on } K \\ 0 & \text{elsewhere} \end{cases}$$

Using the same direct and inverse inequalities as previously yields

$$\begin{aligned} &\|\alpha \operatorname{curl} \mathbf{f}_h + v \operatorname{curl} \mathbf{u}_{\varepsilon h} - v z_{\varepsilon h} - \alpha \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h}\|_{L^2(K)} \\ &\quad \leqslant c((1 + \varepsilon^{1/2} h_K^{-1})\|z_\varepsilon - z_{\varepsilon h}\|_{\varepsilon,K} + \|\mathbf{u}_\varepsilon \cdot \nabla z_\varepsilon - \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h}\|_{L^2(K)} \\ &\quad\quad + \|\operatorname{curl}(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})\|_{L^2(K)} + \|\operatorname{curl}(\mathbf{f} - \mathbf{f}_h)\|_{L^2(K)}) \end{aligned}$$

To bound the nonlinear term, we write

$$\begin{aligned} \|\mathbf{u}_\varepsilon \cdot \nabla z_\varepsilon - \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h}\|_{L^2(K)} \leqslant{}& \|\mathbf{u}_\varepsilon\|_{L^\infty(\Omega)^2} \|\mathbf{grad}(z_\varepsilon - z_{\varepsilon h})\|_{L^2(K)^2} \\ &+ \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{L^2(K)^2} \|\mathbf{grad}\, z_{\varepsilon h}\|_{L^\infty(K)^2} \end{aligned}$$

whence, using (22) and an inverse inequality for $\|\mathbf{grad}\, z_{\varepsilon h}\|_{L^\infty(K)^2}$, we have

$$\|\mathbf{u}_\varepsilon \cdot \nabla z_\varepsilon - \mathbf{u}_{\varepsilon h} \cdot \nabla z_{\varepsilon h}\|_{L^2(K)} \leqslant c(\mathbf{f})\varepsilon^{-1/2}(\|z_\varepsilon - z_{\varepsilon h}\|_{\varepsilon,K} + h_K^{-1}\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{L^2(K)^2})$$

Combining all these and multiplying by $\varepsilon^{-1/2}h_K$ leads to

$$\begin{aligned}&\varepsilon^{-1/2}h_K\|\alpha\,\mathrm{curl}\,\mathbf{f}_h + v\,\mathrm{curl}\,\mathbf{u}_{\varepsilon h} - vz_{\varepsilon h} - \alpha\mathbf{u}_{\varepsilon h}\cdot\nabla z_{\varepsilon h}\|_{L^2(K)}\\ &\quad\leqslant c((1+\varepsilon^{-1}h_K)\|z_\varepsilon - z_{\varepsilon h}\|_{\varepsilon,K} + \varepsilon^{-1}\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(K)^2} + \varepsilon^{-1/2}h_K\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(K)})\end{aligned}$$

Next, for any edge in $\mathscr{E}_K^0$ which is also an edge of another triangle K' of $\mathscr{T}_h$, we take t in (78) equal to

$$t_e = \begin{cases}\mathscr{R}_{e,K}([\partial_{n_K} z_{\varepsilon h}]_e\psi_e) & \text{on } K\\ \mathscr{R}_{e,K'}([\partial_{n_K} z_{\varepsilon h}]_e\psi_e) & \text{on } K'\\ 0 & \text{elsewhere}\end{cases}$$

The same arguments as in the proof of Proposition 5.5, combined with the previous estimate, lead to the desired result.

Since ω_K is the union of at most four elements of $\mathscr{T}_h$, both estimates (74) and (79) are local. When comparing them with (71), we see that the finite element error

$$E_h = \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon$$

is equivalent, up to some terms only involving the data $\mathbf{f}$, to the Hilbertian sum

$$\eta_h = \left(\sum_{K\in\mathscr{T}_h}(\eta_{K\sharp}^2 + \eta_{K\flat}^2)\right)^{1/2} \tag{80}$$

with one of the equivalence constants only depending on U_ε^* and the other one equal to ε^{-1} times a constant only depending on U_ε^*. So these estimates are fully optimal concerning the dependence with respect to h, but they are not optimal with respect to ε (which is not very surprising).

5.3. Conclusions

To conclude, let us introduce the complete error

$$\begin{aligned}E_{\varepsilon h} = {}&\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{(H^1(\Omega)\cap L^\infty(\Omega))^2} + \|p - p_\varepsilon\|_{L^2(\Omega)} + \|z - z_\varepsilon\|_{L^2(\Omega)}\\ &+\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}\|_{H^1(\Omega)^2} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} + \|z_\varepsilon - z_{\varepsilon h}\|_\varepsilon\end{aligned} \tag{81}$$

We deduce from Propositions 5.1 and 5.3 the bound for $E_{\varepsilon h}$.

Corollary 5.7
If the assumptions of Propositions 5.1 and 5.3 hold, there exists a constant $c_5(U_\varepsilon^*)$ only depending on U_ε^* such that

$$E_{\varepsilon h} \leqslant c_3(U_\varepsilon^*)(\eta_\varepsilon^2 + \varepsilon^{-1}|\log h_{\min}| \\ \times \sum_{K\in\mathscr{T}_h}(\eta_{K\sharp}^2 + \eta_{K\flat}^2 + h_K^2\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^2}^2 + (\varepsilon + h_K^2\varepsilon^{-1})\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(K)}^2))^{1/2} \qquad (82)$$

Even though this estimate is not fully optimal when compared with the converse ones, it provides an explicit bound for the global error, which leads to explicit upper and lower bounds for the energy norm of the exact solution. Moreover, the error indicators η_ε, $\eta_{K\sharp}$ and $\eta_{K\flat}$ are appropriate tools for optimizing the choice of ε when adapting the mesh. A possible strategy for that is recalled from Reference [33, Section 5] in the next section.

6. NUMERICAL ALGORITHMS AND EXPERIMENTS

The numerical experiments below are realized by using the code FreeFem++, see Reference [35]. They rely on the discrete problem (47). However, since this problem is nonlinear, we first describe the iterative algorithm that is used for solving it. Next, some numerical tests are presented, first for some analytic solutions in order to validate the code and check the efficiency of the error indicators, and then in more realistic situations.

6.1. The algorithm for solving the discrete problem

We start from an initial guess z^0 in $\mathbb{Z}_h$ and, for each $n \geqslant 1$, we successively solve the problems (we have omitted the index εh for simplicity):

Find $(\mathbf{u}^n, p^n)$ in $\mathbb{X}_h \times \mathbb{M}_h$ such that

$$\forall \mathbf{v}_h \in \mathbb{X}_h, \quad a(\mathbf{u}^n, \mathbf{v}_h) + b(\mathbf{v}_h, p^n) + A(z^{n-1}; \mathbf{u}^n, \mathbf{v}_h) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} \qquad (83)$$

$$\forall q_h \in \mathbb{M}_h, \quad b(\mathbf{u}^n, q_h) = 0$$

Find z^n in $\mathbb{Z}_h$ such that

$$\forall t_h \in \mathbb{Z}_h, \quad c_\varepsilon(z^n, t_h) + C_{\varepsilon h}(\mathbf{u}^n; z^n, t_h) = \alpha\int_\Omega(\mathrm{curl}\,\mathbf{f})t_h\,\mathrm{d}\mathbf{x} + v\int_\Omega(\mathrm{curl}\,\mathbf{u}^n)t_h\,\mathrm{d}\mathbf{x} \qquad (84)$$

with different approximations of the trilinear form $C(\cdot;\cdot,\cdot)$, that we denote by $C_{\varepsilon h}(\cdot;\cdot,\cdot)$. Note that each of them gives rise to a square linear system. We work with two versions of the algorithm, corresponding to two different choices of the trilinear form $C_{\varepsilon h}(\cdot;\cdot,\cdot)$:

- Galerkin algorithm. In this case, the form $C_{\varepsilon h}(\cdot;\cdot,\cdot)$ simply coincides with $C(\cdot;\cdot,\cdot)$.
- Upwind algorithm. According to the ideas presented in References [36, 37] for instance, we denote by $\mathscr{E}_h^0$ the set of edges of elements of $\mathscr{T}_h$ which are not contained in $\partial\Omega$.

With any edge e in $\mathscr{E}_h^0$, we associate a unit normal vector $\mathbf{n}_e$ directed from a triangle K of $\mathscr{T}_h$ towards another triangle K'. We introduce the mean value $\bar{u}_e$ of the quantity $\mathbf{u}\cdot\mathbf{n}_e$ on e, where $\mathbf{u}$ is a vector field continuous on Ω, and also, for any function t in $L^2(\Omega)$, the jump $[t]_e=\bar{t}_{K'}-\bar{t}_K$, where $\bar{t}_K$ and $\bar{t}_{K'}$ stand for the mean values of t on K and K', respectively. Then, the form $C_{\varepsilon h}(\cdot;\cdot,\cdot)$ is defined by

$$C_{\varepsilon h}(\mathbf{u};z,t)=-\alpha\sum_{e\in\mathscr{E}_h^0} z_e^+[t]_e\int_e \mathbf{u}\cdot\mathbf{n}_e\,\mathrm{d}\tau \tag{85}$$

where z_e^+ denotes the mean value of z on K if $\alpha\bar{u}_e$ is nonnegative, and the mean value of z on K' otherwise. Such a technique is known to enhance the convergence of the algorithm when the transport term dominates, i.e. when $|\alpha|$ is large enough.

We now check the boundedness of the sequence resulting from the Galerkin algorithm.

Lemma 6.1
Let us assume that, for a constant $c(\Omega,\sigma)$ only depending on Ω and the regularity parameter σ of the family $(\mathscr{T}_h)_h$ and a fixed value of a real number $q>2$, the parameters h and ε satisfy

$$c(\Omega,\sigma)|\alpha|\|\mathbf{f}\|_{L^2(\Omega)^2}\varepsilon^{-1}q^{1/2}h^{1-2/q}<1 \tag{86}$$

Then, in the case of the Galerkin method, the sequence $(\mathbf{u}^n,p^n,z^n)_{n\geqslant 1}$ of solutions of problem (83)–(84) is bounded in $\mathscr{Y}_\varepsilon^*$.

Proof
We check successively the boundedness of $(\mathbf{u}^n)_{n\geqslant 1}$, $(z^n)_{n\geqslant 1}$ and $(p^n)_{n\geqslant 1}$.

(1) By taking $\mathbf{v}_h$ equal to $\mathbf{u}^n$ in (83) and noting that $A(z^{n-1};\mathbf{u}^n,\mathbf{u}^n)$ is zero, we obtain

$$c_1(\Omega)\|\mathbf{u}^n\|_{H^1(\Omega)^2}\leqslant\|\mathbf{f}\|_{L^2(\Omega)^2} \tag{87}$$

where $c_1(\Omega)$ stands for the constant of the Poincaré–Friedrichs inequality on Ω.

(2) Taking t_h equal to z^n in (84), gives

$$\|z^n\|_\varepsilon^2+C(\mathbf{u}^n;z^n,z^n)\leqslant c(\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}+\|\mathrm{curl}\,\mathbf{u}^n\|_{L^2(\Omega)})\|z^n\|_{L^2(\Omega)}$$

To evaluate the nonlinear term, we observe that

$$C(\mathbf{u}^n;z^n,z^n)=\frac{\alpha}{2}\int_\Omega \mathbf{u}^n\cdot\nabla(z^n)^2\,\mathrm{d}\mathbf{x}=-\frac{\alpha}{2}\int_\Omega(\mathrm{div}\,\mathbf{u}^n)(z^n)^2\,\mathrm{d}\mathbf{x}$$

Thus, we derive from the second line of (83) that, for any t_h in $\mathbb{M}_h$,

$$C(\mathbf{u}^n;z^n,z^n)=-\frac{\alpha}{2}\int_\Omega(\mathrm{div}\,\mathbf{u}^n)((z^n)^2-t_h)\,\mathrm{d}\mathbf{x}$$

whence

$$|C(\mathbf{u}^n;z^n,z^n)|\leqslant\frac{\sqrt{2}}{2}|\alpha|\|\mathbf{u}^n\|_{H^1(\Omega)^2}\|(z^n)^2-t_h\|_{L^2(\Omega)}$$

We have for any q^*, $1<q^*\leqslant 2$,

$$\inf_{t_h\in\mathbb{M}_h}\|(z^n)^2-t_h\|_{L^2(\Omega)}\leqslant c_2(\sigma)h^{2(1-(1/q^*))}|(z^n)^2|_{W^{1,q^*}(\Omega)}$$

and also, by setting $1/q+1/2=1/q^*$ and using the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$,

$$|(z^n)^2|_{W^{1,q^*}(\Omega)}\leqslant 2\|z^n\|_{L^q(\Omega)}\|\mathbf{grad}\,z^n\|_{L^2(\Omega)^2}\leqslant c_3(\Omega)q^{1/2}\|z^n\|^2_{H^1(\Omega)^2}$$

Combining all this with (87), we obtain

$$\|z^n\|_\varepsilon^2-C_0\|z^n\|_\varepsilon^2\leqslant c\|\mathbf{f}\|_{H(\mathbf{curl},\Omega)}\|z^n\|_\varepsilon \tag{88}$$

where C_0 denotes the quantity in the left-hand side of (86). Since C_0 is <1 by assumption, this leads to the estimate for $\|z^n\|_\varepsilon$.

(3) And finally the bound for $\|p^n\|_{L^2(\Omega)}$ follows from the inf-sup condition (39) combined with (87) and (88).

For a fixed value of ε, the boundedness is only proven for h small enough, which is in agreement with Theorem 4.5. It must be noted that Lemma 6.1 is not sufficient to derive the convergence of the method. Moreover the convergence of the upwind algorithm is much more difficult to establish, even in the simple case of an uncoupled transport equation (see Reference [37]). On the other hand, it follows from the arguments in Reference [30] (see also Reference [20, Chapter IV, Theorem 6.3]) that, if the assumptions of Theorem 4.5 hold, for a fixed value of ε, there exists a neighbourhood of any solution of problem (16) independent of h such that the Newton method with initial guess in this neighbourhood produces a sequence which converges towards this solution. However Newton's algorithm seems too expensive for the present model.

Remark 6.2

The implementation of the upwind method is also slightly more expensive than that of the Galerkin method, for the following reason: Let $\mathbf{a}_i$, $1\leqslant i\leqslant N_h$, be the vertices of the triangles of $\mathscr{T}_h$, and, for $1\leqslant i\leqslant N_h$, let φ_i denote the Lagrange function in $\mathbb{Z}_h$ associated with the node $\mathbf{a}_i$; then, the coefficient $C_{\varepsilon h}(\mathbf{u}^n;\varphi_i,\varphi_j)$ of the matrix associated with the transport term in problem (84) vanishes whenever $\mathbf{a}_i$ and $\mathbf{a}_j$ do not belong to the same triangle in the case of the Galerkin method and only when $\mathbf{a}_i$ and $\mathbf{a}_j$ do not belong to adjacent triangles for the upwind method; so the matrix is less sparse in this last case.

6.2. Validation of the iterative algorithm

From now on, we work with the first choice of the space $\mathbb{X}_h$, i.e. piecewise quadratic discrete velocities. The first tests deal with the academic case of a domain Ω equal to the square $]0,\pi[^2$, with the viscosity v equal to 1 and an analytical solution $(\mathbf{u},p,z)$ of problem (2) given by $\mathbf{u}=\mathbf{curl}\,\psi$ and $z=\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})$, with

$$\psi(x,y)=(y(\pi-y)\sin x)^2,\quad p(x,y)=\cos x\cos(2y) \tag{89}$$

The data $\mathbf{f}$ are computed as a function of α and of this solution.

Table I. Convergence for the Navier–Stokes equations ($\alpha = 0$).

n	2	4	6	8	10
	0.124	0.084	0.066	0.045	0.035
n	12	14	16	18	20
	0.029	0.021	0.020	0.016	0.013

Table II. Convergence of the two algorithms for $\alpha = -1$.

n	2	4	6	8	10
Galerkin	3.322	2.988	2.475	1.644	0.794
Upwind	2.994	2.734	2.665	2.848	3.178
n	12	14	16	18	20
Galerkin	0.594	0.538	0.389	0.243	0.138
Upwind	3.514	3.804	4.009	4.116	4.131

Table III. Convergence of the two algorithms for $\alpha = 0.1$.

n	2	4	6	8	10
Galerkin	0.183	0.066	0.027	0.015	0.011
Upwind	0.385	0.272	0.212	0.236	0.228
n	12	14	16	18	20
Galerkin	0.011	0.011	0.011	0.011	0.011
Upwind	0.228	0.229	0.229	0.229	0.229

Table IV. Convergence of the two algorithms for $\alpha = 1$.

n	2	4	6	8	10
Galerkin	3.528	3.256	2.913	2.503	1.867
Upwind	3.057	2.738	2.527	2.555	2.817
n	12	14	16	18	20
Galerkin	1.059	0.761	0.762	0.663	0.487
Upwind	3.154	3.448	3.682	3.830	3.897

Table V. Convergence of the two algorithms for $\alpha = 10$.

n	2	4	6	8	10
Galerkin	608.0	256.8	77.53	50.68	69.10
Upwind	14.74	21.46	23.21	23.34	23.29
n	12	14	16	18	20
Galerkin	440.4	74.74	163.9	94.62	733.9
Upwind	23.27	23.26	23.36	23.42	23.49

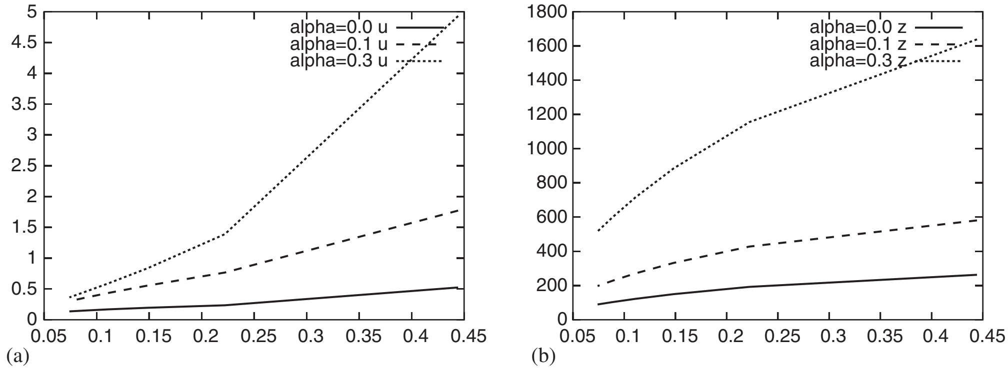


Figure 1. Error for the Galerkin algorithm as a function of h.

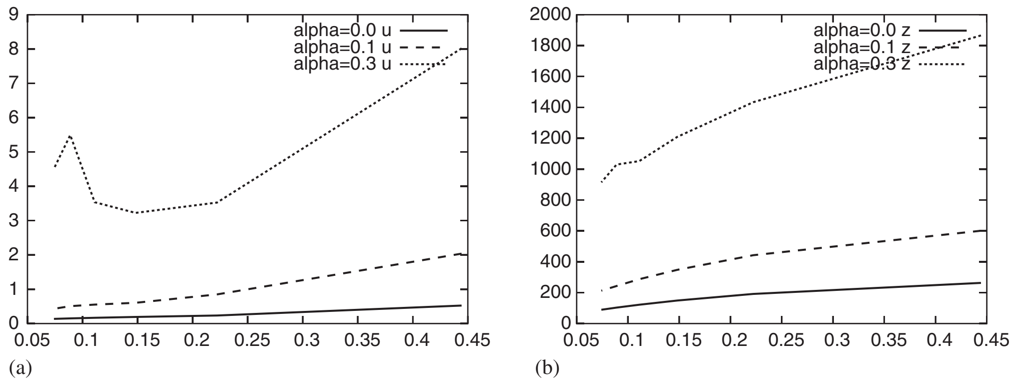


Figure 2. Error for the upwind algorithm as a function of h.

In a first step, we construct a uniform triangulation of Ω into 800 triangles and take

$$h = \frac{\pi\sqrt{2}}{8} \times 10^{-1} \quad \text{and} \quad \varepsilon = 10^{-3}$$

We start with the initial guess $z^0 = 0$, so that problem (83) for $n = 1$ is nothing else than a Stokes problem. Let E^n denote the relative error

$$E^n = \frac{|\mathbf{u} - \mathbf{u}^n|_{H^1(\Omega)^2}}{|\mathbf{u}|_{H^1(\Omega)^2}} + \frac{\|z - z^n\|_{L^2(\Omega)}}{\|z\|_{L^2(\Omega)}}$$

We first consider the case $\alpha = 0$ of the Navier–Stokes equations. Since there is no convection term in (84) in this case, the two algorithms coincide. Table I gives the values of the error E^n for even values of n, $2 \leqslant n \leqslant 20$.

Tables II–V give the values of the error E^n for the same values of n as previously, both for the Galerkin and upwind algorithms, corresponding to the following choices of the parameter α:

$$\alpha = -1, \quad \alpha = 0.1, \quad \alpha = 1, \quad \alpha = 10 \tag{90}$$

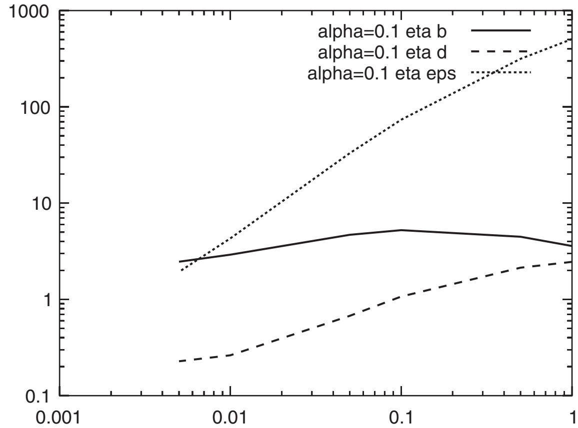


Figure 3. The error indicators as a function of ε.

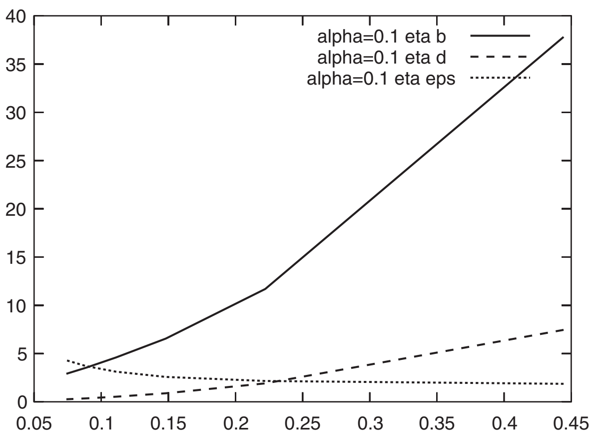


Figure 4. The error indicators as a function of h.

These tables call for two comments:

(i) The convergence rate decreases when $|\alpha|$ increases and at least the Galerkin algorithm does not converge for $\alpha = 10$. This seems in good agreement with Lemma 6.1, see condition (86).

(ii) The convergence of the upwind algorithm seems a little faster than that of the Galerkin algorithm. However the final error E^{20} is larger for the upwind algorithm, which may be due to the lower consistency of the scheme.

6.3. Validation of the discretization method

We work with the same solution $(\mathbf{u}, p, z)$ and the same initial guess z^0 as previously. In view of the poor convergence results for high values of α, we now work with

$$\alpha = 0, \quad \alpha = 0.1, \quad \alpha = 0.3 \tag{91}$$

We take the parameters h and ε such that $h = \varepsilon^{3/2}$, the mesh still being uniform, and perform $n = 10$ iterations in all cases. Figure 1 presents the curves of the relative errors concerning the velocity $\mathbf{u}$ (left part) and the unknown z (right part) for the Galerkin algorithm when h decreases from 0.444 to 0.074. Figure 2 presents the same curves for the upwind algorithm.

The convergence of both methods for the choice $h = \varepsilon^{3/2}$ now seems clear, which is in good agreement with the *a priori* analysis. It can be noted that the error increases with α and also that the error in the approximation of z is very large but does not seem to pollute too much the error in the velocity. It can also be noted that the error is not smaller for the upwind algorithm than for the Galerkin method (and even is larger for $\alpha = 0.3$). Since the implementation of the former is more expensive than for the latter, for reasons explained in Remark 6.2, from now on we work with the Galerkin algorithm.

6.4. Influence of the parameters ε and h

We work with the same solution $(\mathbf{u}, p, z)$, the same initial guess z^0, now with α equal to 0.1 and the Galerkin algorithm.

We construct a uniform triangulation of Ω such that $h = 0.074$. Figure 3 presents in bilogarithmic scales the curves the error indicator η_ε (dotted line) and the Hilbertian sums of the finite element indicators

$$\eta_{h\sharp} = \left(\sum_{K \in \mathcal{T}_h} \eta_{K\sharp}^2 \right)^{1/2} \quad \text{and} \quad \eta_{h\flat} = \left(\sum_{K \in \mathcal{T}_h} \eta_{K\flat}^2 \right)^{1/2} \tag{92}$$

(dashed and plain lines) as a function of ε, for ε decreasing from 1 to 10^{-2}.

Similarly, we fix ε equal to 10^{-2}. Figure 4 presents the curves of η_ε (dotted line), $\eta_{h\sharp}$ (dashed line) and $\eta_{h\flat}$ (plain line) as a function of h, for h decreasing from 0.444 to 0.074.

It can be noted in Figure 4 that the indicator η_ε is nearly independent of h, so at least for this indicator the two types of error are uncoupled. Further computations indicate that these curves are similar to those for other values of α.

6.5. Some real life experiments

In the following simulations, we try to optimize the value of ε when working with adaptive meshes, according to the following strategy (see Reference [33, Section 5] for its first implementation for a different problem).

We first choose a tolerance η^*, we perform a first computation on a quasi-uniform mesh and compute η_ε.

Step 1: If η_ε is smaller than η^*, we go to Step 2. Otherwise, we divide ε by the ratio η_ε/η^* and perform a new computation.

Step 2: We compute the $\eta_{K\sharp}$, the $\eta_{K\flat}$ and their sum η_K, next the mean value $\bar{\eta}_h$ of these η_K. For all K such that η_K is larger than $\bar{\eta}_h$, we divide K into smaller triangles or tetrahedra such that the diameter of these new elements behaves like h_K multiplied by the ratio $\bar{\eta}_h/\eta_K$ (details on the procedure for realizing this can be found in Reference [38, Section 7.5.1]).

Step 2 is iterated 4 or 5 times.

Step 3: We compute η_ε and the Hilbertian sum η_h of the η_K. If η_ε is smaller than η_h, we return to Step 2. Otherwise, we divide ε by a constant times the ratio η_ε/η_h and return to Step 2.

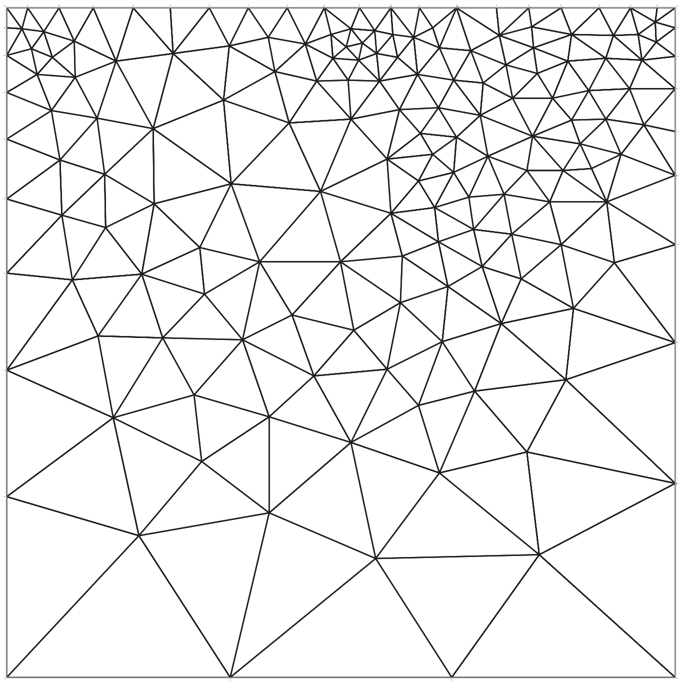


Figure 5. The adapted mesh.

The union of Steps 2 and 3 can also be iterated, at most 3 times in order not to unnecessarily increase the computational cost.

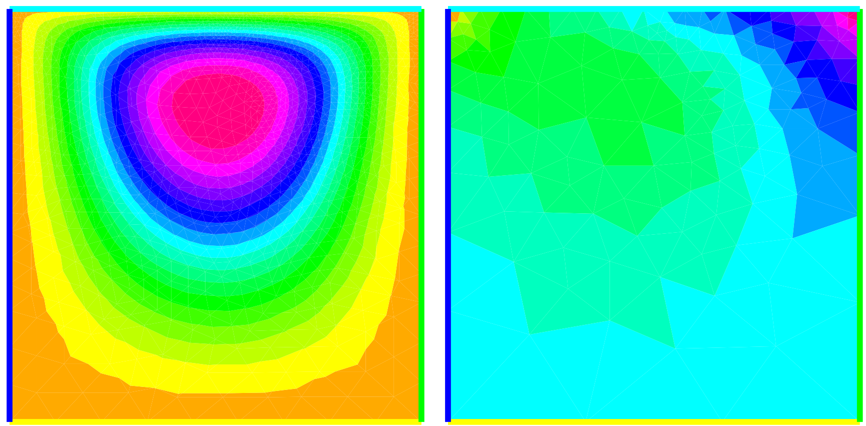
We apply this strategy in the following situation. We work on the unit square $\Omega =]0,1[^2$, with $v = 0.01$ and $\alpha = 0.1$. The data $\mathbf{f}$ here is equal to zero and replaced by the nonhomogenous boundary conditions $\mathbf{g}_b$, corresponding to the so-called regularized driven cavity problem: $\mathbf{g}_b$ is equal to

$$\mathbf{g}_b(x,1) = \begin{pmatrix} 4x(1-x) \\ 0 \end{pmatrix}$$

on the top edge $]0,1[\times \{1\}$ and zero elsewhere. We start with a uniform mesh consisting of 188 triangles ($h = 0.141$) and ε equal to 0.01. We take η^* equal to 0.01 and z^0 equal to zero.

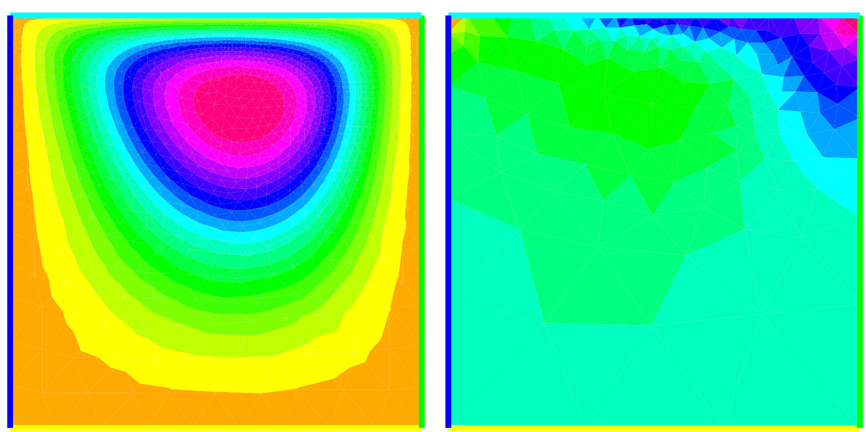
Figure 5 presents the final mesh (295 triangles). The final optimized value of ε is 0.0024. Figure 6 presents the isovalues of the stream function associated with $\mathbf{u}_{\varepsilon h}$ (left part) and of the pressure (right part).

Finally, Figure 7 presents the isovalues of the stream function associated with $\mathbf{u}_{\varepsilon h}$ (left part) and of the pressure (right part) in the case $\alpha = 0$ of the Navier–Stokes equations with the same data. The only purpose of this last computation is to show that, even for small values of α, the flow of the grade-two fluid is different from the flow of a fluid governed by the Navier–Stokes equations.



(a) (b)

Figure 6. Isovalues of the stream function and of the pressure for $\alpha = 0.1$.



(a) (b)

Figure 7. Isovalues of the stream function and of the pressure for $\alpha = 0$.

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